THE PRINCIPLES OF

EQUATION SUB-ELEMENT THEORY

VOLUME THREE OF FOUR

SECTION 24

RELATED PROBLEMS 1 THRU 40

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SECTION 24. *RELATED PROBLEMS.*

Problems presented below appear in sequence by *applicable section* which explains how each is to be resolved. This is intended to allow for easy *cross referencing*.

PROBLEM NUMBER 1 *(Ref. Section 2.1)*

GIVEN:

Equation 4

SOLVE:

Equation 4 for $\theta = 20^{o}$

SOLUTION:

Where,

$$\theta = 20^{o}$$

$$3\theta = 60^{o}$$

$$\tau = \cos(3\theta) = 0.5$$

$$\cos^2\theta + (\frac{2\tau\lambda - 5}{6\lambda})\cos\theta - \frac{\tau}{2\lambda} = 0$$ *[Ref. Equation 4]*

Such that

$$\sin\phi = \frac{1}{2\cos\theta}$$

$$= \frac{1}{2\cos 20^{o}}$$

$$= 0.532088886$$

$$\phi = 32.1467014^{o}$$

$$3\phi = 96.44010419^{o}$$

$$\lambda = \sin(3\phi) = 0.993689653$$

$$\cos^2\theta + [\frac{2(0.5)(0.993689653) - 5}{6(0.993689653)}]\cos\theta - \frac{0.5}{2(0.993689653)} = 0$$

$$\cos^2\theta - 0.671958683\cos\theta - 0.251587605 = 0$$

Checking via Quadratic Formula renders:

$$\cos\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+0.671958683 \pm \sqrt{(-0.671958683)^2 + 4(1)(0.251587605)}}{2(1)}$$

$$= \frac{+0.671958683 \pm 1.207426558}{2}$$

$$= 0.93969262$$ *Q.E.D*

PROBLEM NUMBER 2 *(Ref. Section 2.1)*

GIVEN:

The following *general trigonometric formula*:

$$\sin[2(a\theta)] = 2\sin(a\theta)\cos(a\theta) = \sin(3a\theta - a\theta) = \sin(3a\theta)\cos(a\theta) - \cos(3a\theta)\sin(a\theta)$$

DETERMINE:

Straight line expressions for each of the following:

$1/\sin\theta$ in terms of $1/\cos\theta$ and trigonometric relationships of 3θ

$1/\sin(2\theta)$ in terms of $1/\cos(2\theta)$ and trigonometric relationships of 3θ

$1/\sin(4\theta)$ in terms of $1/\cos(4\theta)$ and trigonometric relationships of 3θ

CONSTRUCT:

A drawing which represents the three *Linearized Expressions* determined above when $\theta = 20^o$

VERIFY:

That $\zeta = \sqrt{3} = 4\sin\theta + \tan\theta$ for $\theta = 20^o$

SOLUTION:

Determination:

For $a = 1$

$$\sin[2(a\theta)] = \sin(2\theta) = 2\sin\theta\cos\theta = \sin(3a\theta - a\theta) = \sin(3\theta - \theta) = \sin(3\theta)\cos\theta - \cos(3\theta)\sin\theta$$

$$\sin(2\theta) = \sin(3\theta - \theta)$$

$$2\sin\theta\cos\theta = \sin(3\theta)\cos\theta - \cos(3\theta)\sin\theta$$

$$2 = \frac{\sin(3\theta)\cos\theta}{\sin\theta\cos\theta} - \frac{\cos(3\theta)\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin(3\theta)}{\sin\theta} - \frac{\cos(3\theta)}{\cos\theta}$$

$$-\frac{\sin(3\theta)}{\sin\theta} = -2 - \frac{\cos(3\theta)}{\cos\theta}$$

$$\frac{\sin(3\theta)}{\sin\theta} = 2 + \frac{\cos(3\theta)}{\cos\theta}$$

$$\frac{1}{\sin\theta} = \frac{1}{\tan(3\theta)}(\frac{1}{\cos\theta}) + \frac{2}{\sin(3\theta)}$$

$$y = mx + b$$

Or,

$$y = \frac{1}{\sin\theta}$$

$$m = \frac{1}{\tan(3\theta)}$$

$$b = \frac{2}{\sin(3\theta)}$$

For $a = 2$

$$\sin[2(a\theta)] = \sin(4\theta) = 2\sin(2\theta)\cos(2\theta) = \sin(3a\theta - a\theta) = \sin(6\theta - 2\theta) = \sin(6\theta)\cos(2\theta) - \cos(6\theta)\sin(2\theta)$$

$$\sin(4\theta) = \sin(6\theta - 2\theta)$$

$$2\sin(2\theta)\cos(2\theta) = \sin(6\theta)\cos(2\theta) - \cos(6\theta)\sin(2\theta)$$

$$2 = \frac{\sin(6\theta)}{\sin(2\theta)} - \frac{\cos(6\theta)}{\cos(2\theta)}$$

$$-\frac{\sin(6\theta)}{\sin(2\theta)} = -2 - \frac{\cos(6\theta)}{\cos(2\theta)}$$

$$\frac{1}{\sin(2\theta)} = \frac{1}{\tan(6\theta)}[\frac{1}{\cos(2\theta)}] + \frac{2}{\sin(6\theta)}$$

$$y = mx + b$$

Or,

$$y = \frac{1}{\sin(2\theta)}$$

$$m = \frac{1}{\tan(6\theta)}$$

$$b = \frac{2}{\sin(6\theta)}$$

For $a = 4$

$$\sin[2(a\theta)] = \sin(8\theta) = 2\sin(4\theta)\cos(4\theta) = \sin(3a\theta - a\theta) = \sin(12\theta - 4\theta) = \sin(12\theta)\cos(4\theta) - \cos(12\theta)\sin(4\theta)$$

$$\sin(8\theta) = \sin(12\theta - 4\theta)$$

$$2\sin(4\theta)\cos(4\theta) = \sin(12\theta)\cos(4\theta) - \cos(12\theta)\sin(4\theta)$$

$$2 = \frac{\sin(12\theta)}{\sin(4\theta)} - \frac{\cos(12\theta)}{\cos(4\theta)}$$

$$-\frac{\sin(12\theta)}{\sin(4\theta)} = -2 - \frac{\cos(12\theta)}{\cos(4\theta)}$$

$$\frac{1}{\sin(4\theta)} = \frac{1}{\tan(12\theta)}[\frac{1}{\cos(4\theta)}] + \frac{2}{\sin(12\theta)}$$

$$y = mx + b$$

Or,

$$y = \frac{1}{\sin(4\theta)}$$

$$m = \frac{1}{\tan(12\theta)}$$

$$b = \frac{2}{\sin(12\theta)}$$

Construction:

For

$$\theta = 20^o$$
$$3\theta = 60^o$$

$$\sin(3\theta) = \sin 60^o \ = \eta = \sqrt{3}/2$$

$$\cos(3\theta) = \cos 60^o \ = \tau = 1/2$$

$$\sin(6\theta) = \sin(2x3\theta) = 2\sin(3\theta)\cos(3\theta)$$
$$= 2\eta\tau$$
$$= \eta$$

$$\sin(12\theta) = \sin(2x6\theta) = 2\sin(6\theta)\cos(6\theta)$$
$$= 2\eta(-\tau)$$
$$= -\eta$$

$$\tan(3\theta) = \tan 60^o \ = \zeta = \sqrt{3}$$

$$\tan(6\theta) = \tan(2x3\theta)$$
$$= \frac{2\tan(3\theta)}{1-\tan(3\theta)^2}$$
$$= \frac{2\zeta}{1-\zeta^2}$$
$$= \frac{2\zeta}{1-3}$$
$$= -\zeta$$

$$\tan(12\theta) = \tan(2x6\theta)$$
$$= \frac{2\tan(6\theta)}{1-\tan(6\theta)^2}$$
$$= \frac{2(-\zeta)}{1-\zeta^2}$$
$$= -\frac{2\zeta}{1-3}$$
$$= \zeta$$

So,

$$\frac{1}{\sin\theta} = \frac{1}{\tan(3\theta)}(\frac{1}{\cos\theta}) + \frac{2}{\sin(3\theta)} = \frac{1}{\zeta}(\frac{1}{\cos\theta}) + \frac{2}{\eta}$$

$$\frac{1}{\sin(2\theta)} = \frac{1}{\tan(6\theta)}[\frac{1}{\cos(2\theta)}] + \frac{2}{\sin(6\theta)} = -\frac{1}{\zeta}[\frac{1}{\cos(2\theta)}] + \frac{2}{\eta}$$

$$\frac{1}{\sin(4\theta)} = \frac{1}{\tan(12\theta)}[\frac{1}{\cos(4\theta)}] + \frac{2}{\sin(12\theta)} = \frac{1}{\zeta}[\frac{1}{\cos(4\theta)}] - \frac{2}{\eta}$$

Each of these three straight line equations is depicted in *Figure 61*, where:

$$\frac{1}{\sin\theta}=\frac{1}{\zeta}\left(\frac{1}{\cos\theta}\right)+\frac{2}{\eta} \quad \textit{Constructed as Euclidean straight line } \overline{AC} \textit{ extended}$$

$$\frac{1}{\sin(2\theta)}=-\frac{1}{\zeta}\left[\frac{1}{\cos(2\theta)}\right]+\frac{2}{\eta} \quad \textit{Constructed as Euclidean straight line } \overline{AD} \textit{ extended}$$

$$\frac{1}{\sin(4\theta)}=\frac{1}{\zeta}\left[\frac{1}{\cos(4\theta)}\right]-\frac{2}{\eta} \quad \textit{Constructed as Euclidean straight line } \overline{BF} \textit{ extended}$$

Where,

- $\overline{LG}$ is perpendicular to line $\overline{OC}$
- $\overline{KH}$ is perpendicular to line $\overline{OD}$
- $\overline{JI}$ is perpendicular to line $\overline{OF}$

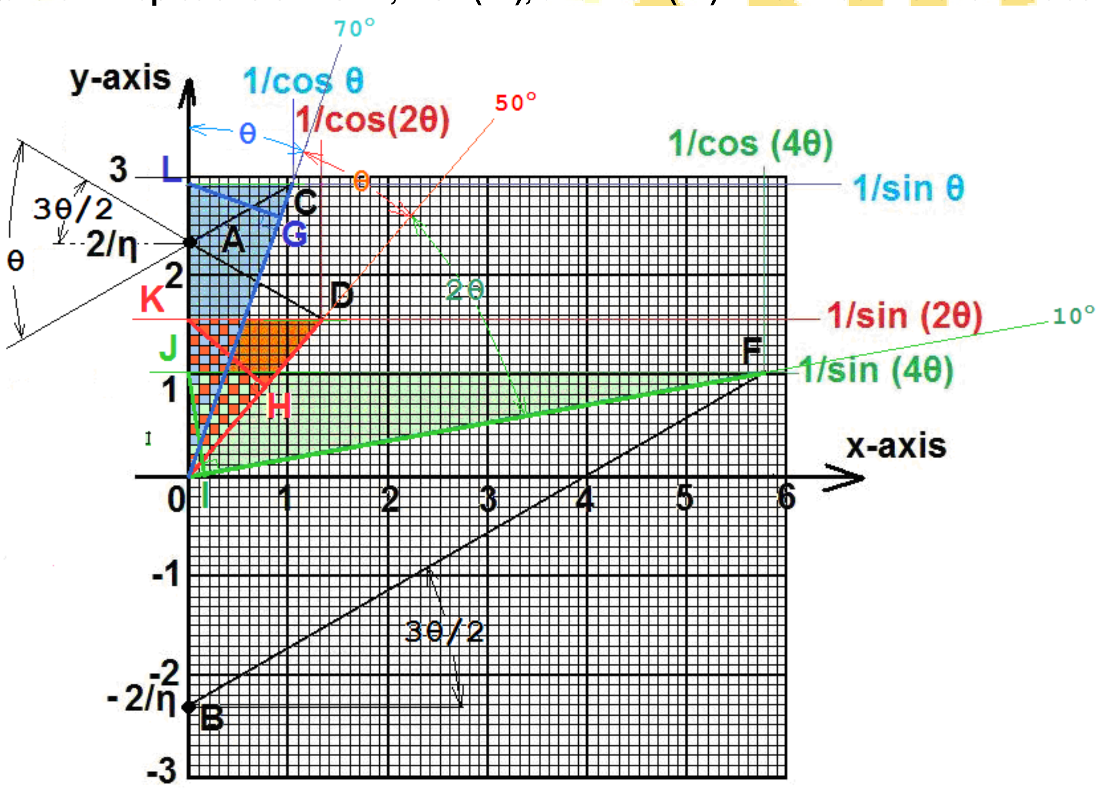
From the geometry in the figure,

$$\overline{LG}=\overline{KH}=\overline{JI}=1$$

In this case, the respective y-intercepts for the straight-lines indicated above are of the following values:

$$\pm\frac{2}{\eta}=\pm\frac{2}{\sqrt{3}/2}=\pm\frac{4}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)=\pm\frac{4}{3}\sqrt{3}=\pm 2.309401077$$

Figure 61. Depictions of 1/sinθ, 1/sin(2θ), and 1/sin(4θ) *Linearizations of the Cube*.



Verification:

Where:

$$\sin(2\theta) = 2\sin\theta\cos\theta = \sin(3\theta-\theta)$$
$$= \sin(3\theta)\cos\theta - \cos(3\theta)\sin\theta$$
$$= \eta\cos\theta - \tau\sin\theta$$

$$4\sin\theta\cos\theta = (2\eta)\cos\theta - (2\tau)\sin\theta$$
$$= (2\frac{\sqrt{3}}{2})\cos\theta - (2\frac{1}{2})\sin\theta$$
$$= \sqrt{3}\cos\theta - \sin\theta$$
$$= \zeta\cos\theta - \sin\theta$$

Then,

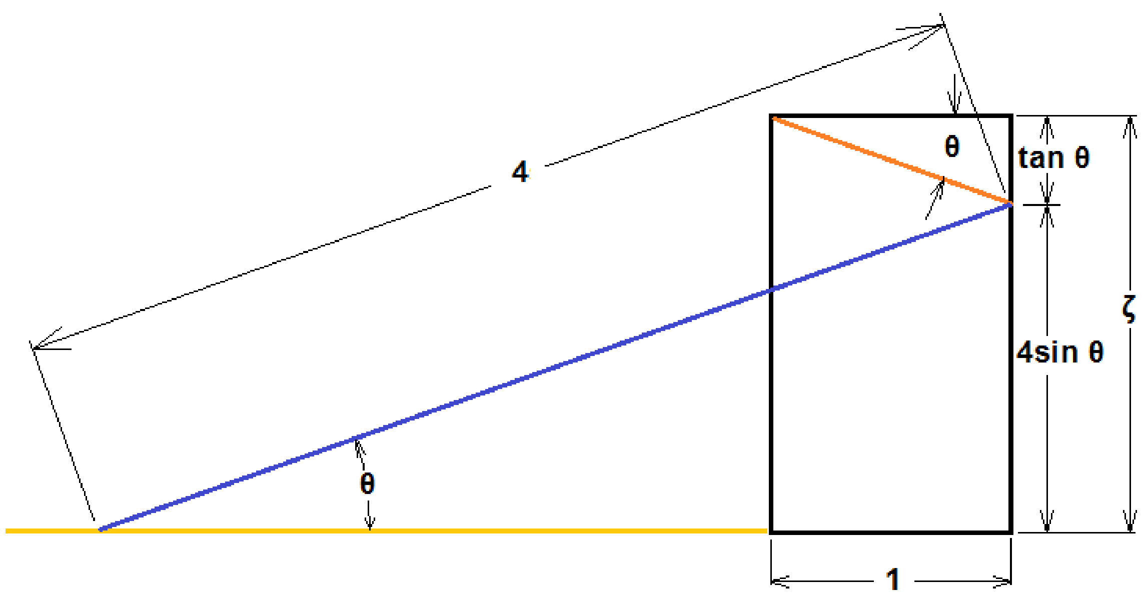
$$4\sin\theta\cos\theta + \sin\theta = \zeta\cos\theta$$
$$\frac{4\sin\theta\cos\theta + \sin\theta}{\cos\theta} = \zeta$$

Or,

$$4\sin\theta + \tan\theta = \zeta \qquad \text{Q.E.D.}$$

This relationship is shown in *Figure 62*.

Figure 62. Illustration of 4sin θ + tan θ = ζ for 3θ=60°.



Lastly, since alternate angles are equal, notice that *Figure 62* depicts rays whose *angles of incidence* and *reflection* both are exactly equal to θ°. Hence such figure can be used to analyze physical wave motion.

PROBLEM NUMBER 3 *(Ref. Section 2.2)*

GIVEN:

The following two *Quadratic Equations*:

$$1.4x^2 + 16x + 12 = 0$$

$$x^2 - 11x + 10.662 = 0$$

DEVELOP:

A *Complex Quadratic Equation* which encompasses all *given Quadratic Equation* roots

SOLUTION:

Letting,

$$1.4x_1^2 + 16x_1 + 12 = 0$$

$$x_2^2 - 11x_2 + 10.662 = 0$$

The *Quadratic Formula* calculates roots for the two above *Quadratic Equations* as follows:

$$1.4x_1^2 + 16x_1 + 12 = 0$$

$$x_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-16 \pm \sqrt{(16)^2 - 4(1.4)(12)}}{2(1.4)}$$

$$= \frac{-16 \pm \sqrt{(256) - 67.2}}{2.8}$$

$$= \frac{-16 \pm \sqrt{188.8}}{2.8}$$

$$= \frac{-16 \pm 13.74045123}{2.8}$$

$$x_{1a}; x_{1b} = -0.806981703, -10.62158973$$

$$x_2^2 - 11x_2 + 10.662 = 0$$

$$x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+11 \pm \sqrt{(-11)^2 - 4(1)(10.662)}}{2(1)}$$

$$= \frac{11 \pm \sqrt{(121) - 42.648}}{2}$$

$$= \frac{11 \pm \sqrt{78.352}}{2}$$

$$= \frac{11 \pm 8.85166651}{2}$$

$$x_{2a}; x_{2b} = 9.925833255, 1.074166745$$

Each of the two given equations sum to zero; hence, they are equal to each other. They are equated in order to produce a single *Complex Quadratic Equation* as follows:

$$1.4x_1^2 + 16x_1 + 12 = x_2^2 - 11x_2 + 10.662$$

$$1.4x_1^2 - x_2^2 + 16x_1 + 11x_2 + 1.338 = 0$$

Substituting respective x_{1a} and x_{2a} roots above into this new *Complex Quadratic Equation* gives:

$$1.4x_1^2 - x_2^2 + 16x_1 + 11x_2 + 1.338 = 0$$

$$1.4(-0.806981703)^2 - (9.925833255)^2 + 16(-0.806981703) + 11(9.925833255) + 1.338 = 0$$

$$0.911707257 - 98.52216581 - 12.91170725 + 109.1841658 + 1.338 = 0$$

$$111.4338731 + 111.4338731 = 0$$

Substituting x_{1a} and x_{2b} roots yields:

$$1.4x_1^{\;2} - x_2^{\;2} + 16x_1 + 11x_2 + 1.338 = 0$$
$$1.4(-0.806981703)^2 - (1.074166745)^2 + 16(-0.806981703) + 11(1.074166745) + 1.338 = 0$$
$$0.911707257 - 1.153834196 - 12.91170725 + 11.8158342 + 1.338 = 0$$
$$14.06554145 - 14.06554145 = 0$$

Substituting respective x_{1b} and x_{2b} roots above gives:

$$1.4x_1^{\;2} - x_2^{\;2} + 16x_1 + 11x_2 + 1.338 = 0$$
$$1.4(-10.62158973)^2 - (1.074166745)^2 + 16(-10.62158973) + 11(1.074166745) + 1.338 = 0$$
$$157.9454357 - 1.153834196 - 169.9454357 + 11.8158342 + 1.338 = 0$$
$$171.0992699 - 171.0992699 = 0$$

Lastly, substituting x_{1b} and x_{2a} roots yields:

$$1.4x_1^{\;2} - x_2^{\;2} + 16x_1 + 11x_2 + 1.338 = 0$$
$$1.4(-10.62158973)^2 - (9.925833255)^2 + 16(-10.62158973) + 11(9.925833255) + 1.338 = 0$$
$$157.9454357 - 98.52216581 - 169.9454357 + 109.1841658 + 1.338 = 0$$
$$268.4676015 - 268.4676015 = 0$$

As indicated, substitution of roots in any order satisfies the *Complex Quadratic Equation*:

$$1.4x_1^{\;2} - x_2^{\;2} + 16x_1 + 11x_2 + 1.338 = 0$$

Now, note that substituting $x_2 = 0$ into the *Complex Quadratic Equation* yields:

$$1.4x_1^{\;2} - x_2^{\;2} + 16x_1 + 11x_2 + 1.338 = 0$$
$$1.4x_1^{\;2} - (0)^2 + 16x_1 + 11(0) + 1.338 = 0$$
$$1.4x_1^{\;2} + 16x_1 + 1.338 = 0$$

Likewise, substituting $x_1 = 0$ into the *Complex Quadratic Equation* renders:

$$1.4x_1^{\;2} - x_2^{\;2} + 16x_1 + 11x_2 + 1.338 = 0$$
$$1.4(0)^2 - x_2^{\;2} + 16(0) + 11x_2 + 1.338 = 0$$
$$x_2^{\;2} - 11x_2 - 1.338 = 0$$

These two respective results do not match the given *Quadratic Equations* and, hence, do not yield their same respective root sets.

This is because they characterize *additional root sets* for the new *Complex Quadratic Equation* which clearly do not satisfy the given *Quadratic Equations*.

PROBLEM NUMBER 4 *(Ref. Section 2.2)*

GIVEN:

Equation 8 and *Equation 9*

DEVELOP:

A *Complex Quadratic Equation* in terms of y_1, y_2, and η using just the *above given equations*

SOLUTION:

Where,

$$y_1 + y_2 + y_3 = 0$$ *[Ref. Equation 9]*

$$y_3 = -(y_1 + y_2)$$

Via substitution,

[Ref. Equation 8]

$$y_1 y_2 y_3 = -\frac{\eta}{4}$$

$$-y_1 y_2 (y_1 + y_2) = -\frac{\eta}{4}$$

Or,

$$y_1^2 y_2 + y_1 y_2^2 = \frac{\eta}{4}$$

Where

$$y_1 = \sin\theta$$

$$y_2 = \sin(\theta + 120^o)$$

PROBLEM NUMBER 5 *(Ref. Section 2.3)*

GIVEN:

The following *Quadratic Equation*:

$$3/4x^2 + 7x + 57/4 = 0$$

CONSTRUCT:

x:

SOLUTION:

Where,

$$t = \sqrt{b^2 - 4ac} = \sqrt{7^2 - 4(3/4)(57/4)}$$
$$= \sqrt{49(4/4) - (3)(57/4)}$$
$$= \sqrt{(196 - 171)/4}$$
$$= \sqrt{25/4}$$
$$= 5/2$$

$$\Delta = b - t$$
$$= 7 - 5/2$$
$$= 9/2$$

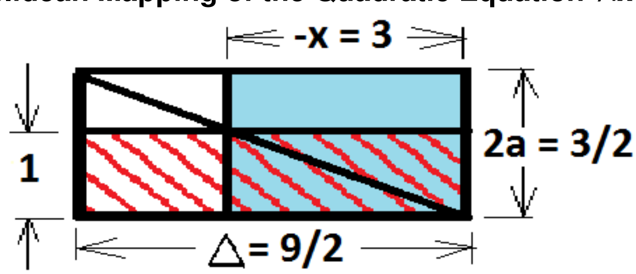
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-b \pm t}{2a}$$
$$= \frac{-\Delta}{2a}; \frac{-(b+t)}{2a}$$

Or,

$$\frac{-x}{1} = \frac{+\Delta}{2a}$$
$$= \frac{9/2}{3/2}$$
$$= 3$$

The necessary construction is afforded in *Figure 63*:

Figure 63. Euclidean Mapping of the Quadratic Equation ¾x² + 7x + 57/4 = 0.



Problem Number 6 *(Ref. Section 2.3)*

GIVEN:

With respect to *Figure 1*:

- $a = b = 3t/4 = 2$ *Units of Measurement*
- $\theta = 30^o$
- $\mathrm{s} = 2\mathrm{r}$

DETERMINE:

a) The associated *Quadratic Equation*
b) Its first root, x_1, via *Euclidean Construction*

SOLUTION:

a) Where,

$$\frac{3}{4}t = 2$$

$$t = \frac{8}{3} = \sqrt{b^2 - 4ac}$$

$$t^2 = \frac{64}{9} = b^2 - 4ac$$

$$= (2)^2 - 4(2)c$$

$$\frac{64}{36} = 1 - 2c$$

$$2c = \frac{36 - 64}{36}$$

$$c = \frac{-28}{72}$$

$$= -\frac{7}{18}$$

The associated *Quadratic Equation* is:

$$ax^2 + bx + c = 0$$

$$2x^2 + 2x - \frac{7}{18} = 0$$

b) The upper illustration of *Figure 64* expresses *Figure 1 geometry.* The *lower one* designates its associated Euclidean construction. Pertaining to the latter, the specifics of such construction are as follows:

- Circle O′ has its center at Point O and has a radius of r
- Circle C′ has its center at Point C and has a radius of r
- Their upper intersection designates Point B such that $\overline{OC} = \overline{OB} = \overline{BC} = r$
- Right triangle ABC shows its 90° vertex lying upon the circumference of circle O′ with its hypotenuse stretching across its diameter
- The 30°, 60° right triangle ABC possesses a shorter side which is equal in length to one-half the length of its hypotenuse
- Point E lies at the bottom of the altitude of equilateral triangle OBC and therefore divides side OC in half. Therefore:
  - $\overline{EC} = r/2$
  - $\overline{AE} = 3\,r/2$
- Line $\overline{DE}$ is drawn parallel to line $\overline{BC}$ as follows:
  - Circle E′ has its center at Point E and is of sufficient radius to cross line AB at two locations, designated as Points U and V, respectively
  - Circles U′ and V′ have their respective centers located at Points U and V, the intersection points of circle E′ with line $\overline{AB}$, and are of equal arbitrary radii
  - Point D is designated at the intersection of line AB with a line which is drawn from Point E and passes thru the two intersection points of circles U′ and V′, respectively
- 30°, 60° right triangle ADE is similar to 30°, 60° right triangle ABC and, therefore, has similar respective sides

  Hence,

$$\frac{\overline{AD}}{\overline{AE}} = \frac{\overline{AB}}{\overline{AC}},$$

or,

$$\frac{\overline{AD}}{\frac{3r}{2}} = \frac{\overline{AB}}{2r}$$

$$\frac{2\overline{AD}}{3r} = \frac{\overline{AB}}{2r}$$

$$4r\overline{AD} = 3r\overline{AB}$$

$$\overline{AD} = \frac{3}{4}\overline{AB}$$

Recalling that,

$$t = \frac{8}{3}$$

$$\Delta = t - b$$

$$= \frac{8}{3} - 2$$

$$= \frac{2}{3}$$

Furthermore,

$$\frac{\overline{AE}}{b} = \frac{\overline{AC}}{t}$$

$$\frac{3\,\mathrm{r}/2}{b} = \frac{3\,\mathrm{r}/2 + \mathrm{r}/2}{t}$$

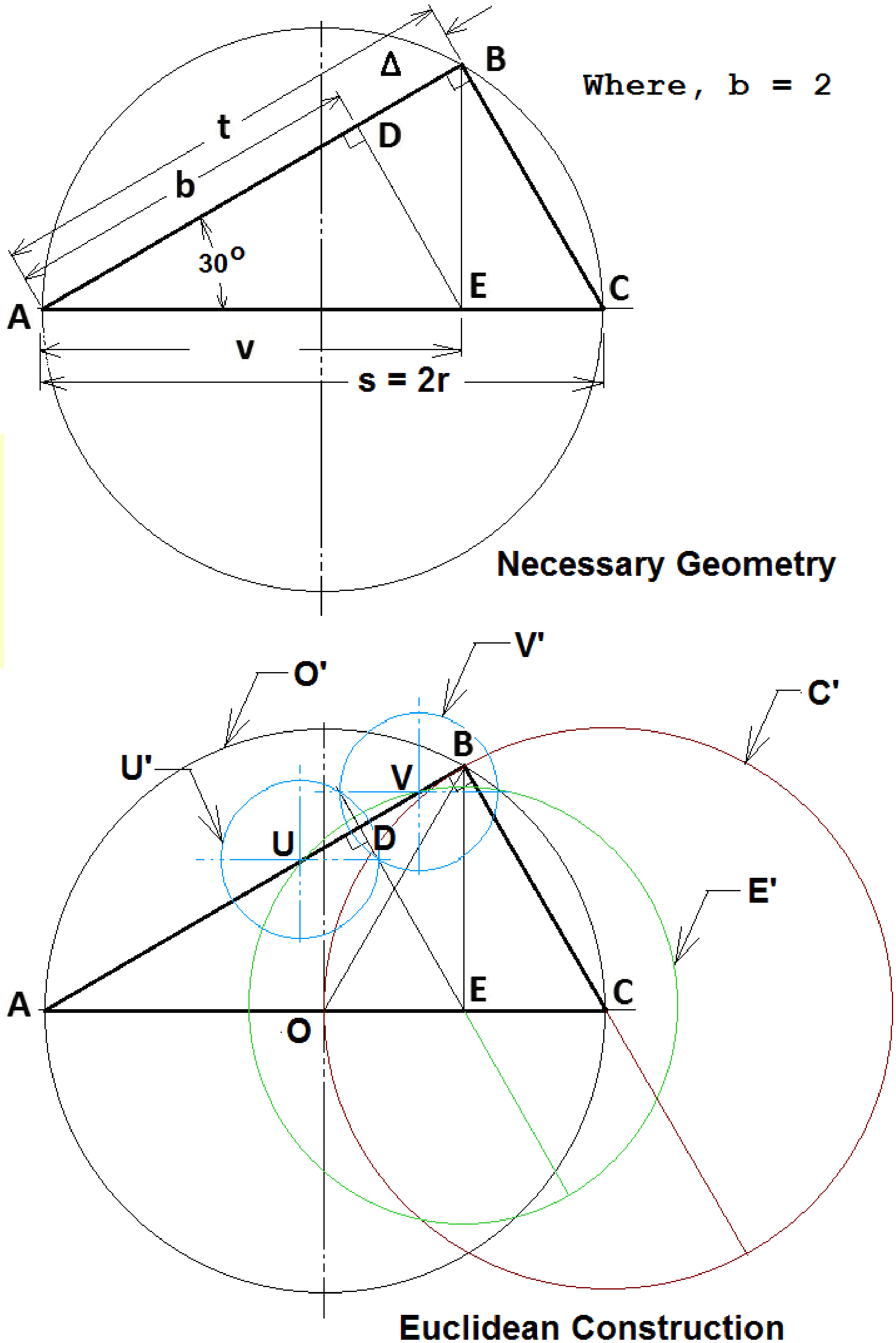
$$\frac{3\,\mathrm{r}/2}{b} = \frac{4\,\mathrm{r}/2}{t}$$

$$\frac{3}{b} = \frac{4}{t}$$

Confirming that,

$$3t/4 = b$$

Figure 64. 30° Right. Triangle Construction with ‘b’ Equal to ¾ of Length ‘t’.



The completion of this Euclidean construction is portrayed in *Figure 65* such that:

- Circle D′ has its center at Point D and has a radius of $b=2$
- Point F is located on circle D′ where it intersects line DE extended
- Circle F′ has its center at Point F and also has a radius of $b=2$
- Point G is located on circle F′ where it intersects line $\overline{DF}$ extended
- The straight line diagonal $\overline{BG}$ is then drawn
- The offset of 1 unit is constructed as the perpendicular bisector of line DF where Points H and J represent the respective intersection points between circles D′ and F′
- Point K represents the intersection point between diagonal $\overline{BG}$ and straight line $\overline{JH}$
- Straight line $\overline{BM}$ is set off parallel to and the same length as line $\overline{DG}$
- Point L designates the intersection point between straight lines $\overline{BM}$ and $\overline{JH}$
- Right triangles BGM and BKL contain the common angle GBM = angle KBL
- Hence, triangles BGM and BKL are similar and have respective similar sides such that,

$$\frac{\overline{KL}}{1} = \frac{x_1}{1} = \frac{\Delta}{2a}$$

$$= \frac{2/3}{2(2)}$$

$$x_1 = \frac{1}{6}$$

Check,

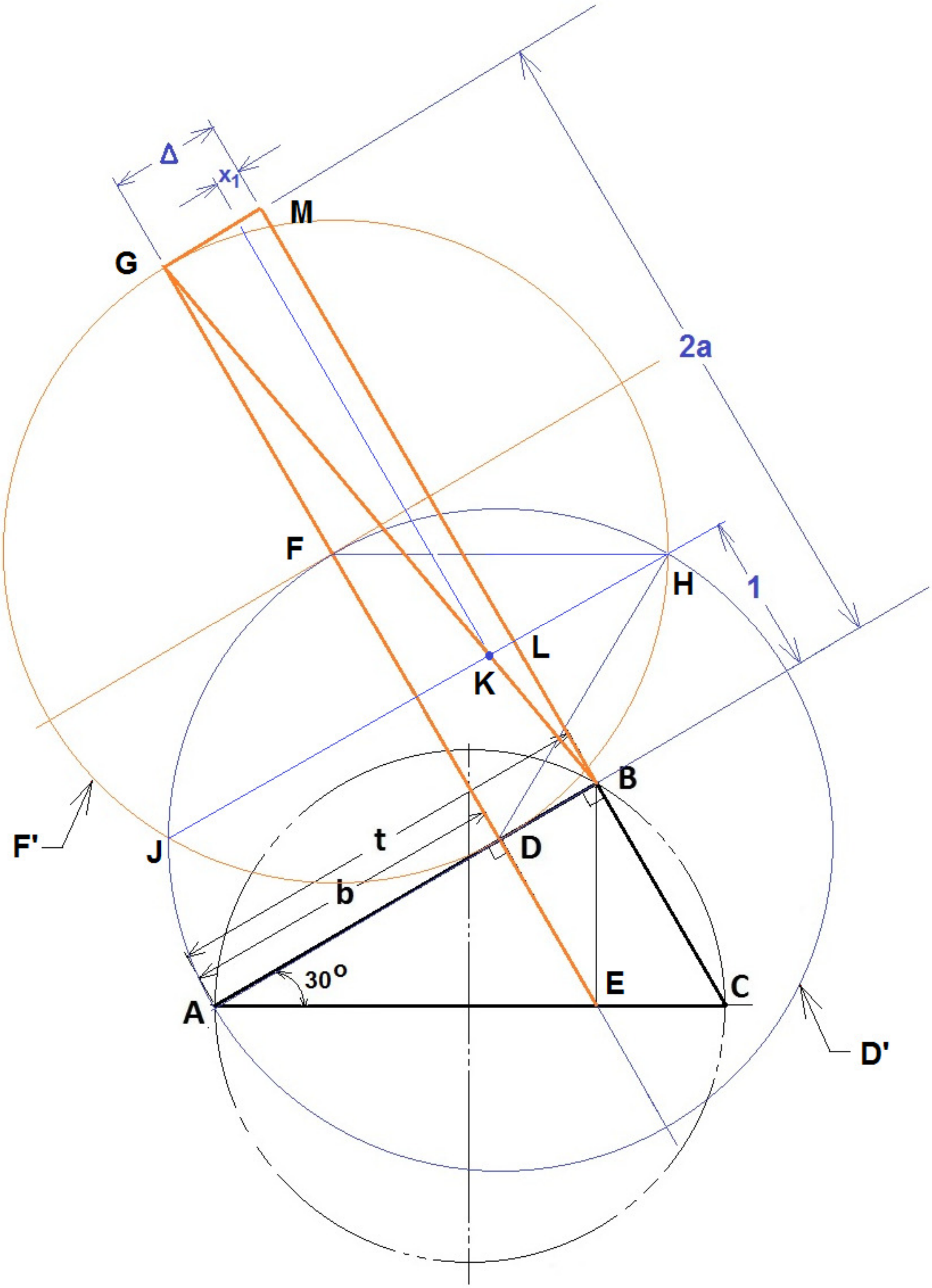
$$2x^2 + 2x - \frac{7}{18} = 0$$

$$2(\frac{1}{6})^2 + 2(\frac{1}{6}) - \frac{7}{18} = 0$$

$$\frac{1}{18} + \frac{6}{18} - \frac{7}{18} = 0$$

$$0 = 0$$

Figure 65. Resulting Euclidean Construction of Lengths ‘2a’ and ‘x1’.



PROBLEM NUMBER 7 *(Ref. Section 2.4.3)*

GIVEN:

Equation 11 as follows:

$$z_1 z_2 z_3 = -\zeta \qquad \text{[Ref. Equation 11]}$$

Where,

$$z_1 = \tan\theta$$
$$z_2 = \tan\theta + 120^o$$
$$z_3 = \tan\theta + 240^o$$

CONSTRUCT:

A geometry which relates the above given relationship to a right triangle when $3\theta = 60^o$

SOLUTION:

Where

$$3\theta = 60^o$$
$$\theta = 20^o$$
$$z_1 = \tan\theta = \tan 20^o$$

$$\theta + 120^o = 20^o + 120^o$$
$$= 140^o$$
$$= 180^o - 40^o$$
$$= 180^o - 2(20^o)$$
$$= 180^o - 2\theta$$
$$z_2 = \tan(180^o - 2\theta)$$
$$= -\tan 2\theta$$

$$\theta + 240^o = 20^o + 240^o$$
$$= 260^o$$
$$= 180^o + 80^o$$
$$= 180^o + 4(20^o)$$
$$= 180^o + 4\theta$$
$$z_3 = \tan(180^o + 4\theta)$$
$$= \tan 4\theta$$

Then,

$$z_1 z_2 z_3 = -\zeta$$
$$-\tan\theta \tan(2\theta)\tan(4\theta) = -\zeta$$
$$\tan\theta \tan(2\theta)\tan(4\theta) = \zeta$$

Figure 66 demonstrates this affinity as follows:

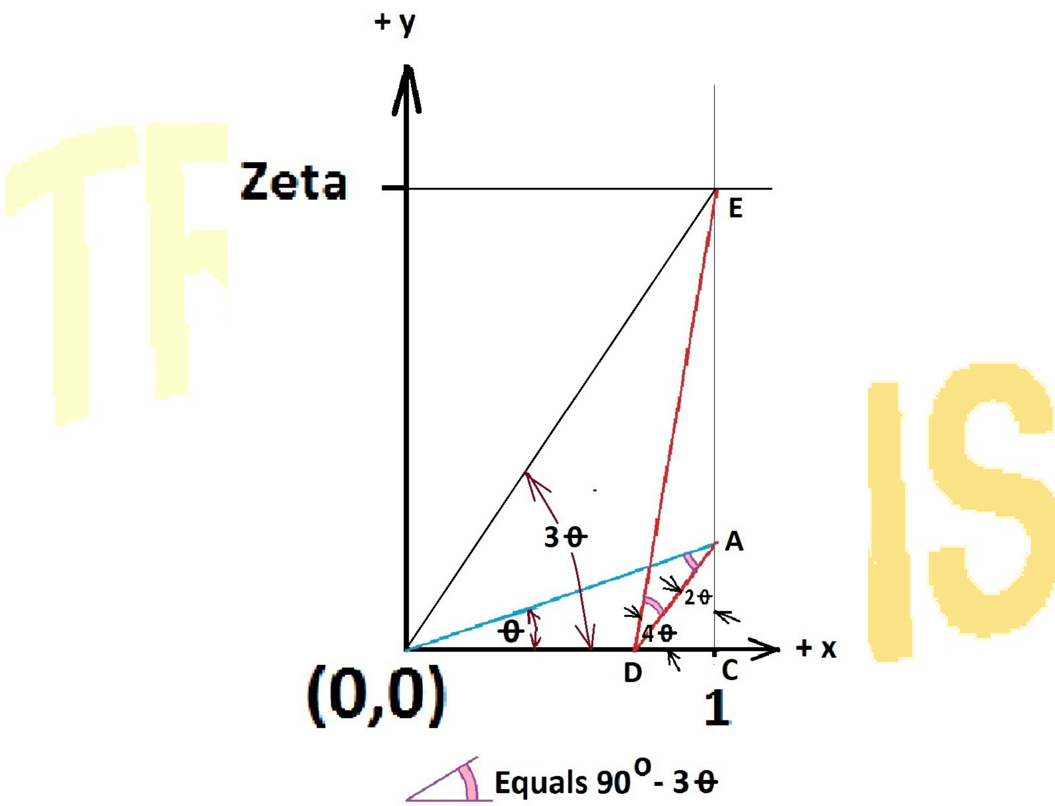
Where,

$$\overline{AC} = \tan\theta$$

$$\overline{CD} = \overline{AC}\tan(2\theta) = \tan\theta\tan(2\theta)$$

$$\overline{CE} = \overline{CD}\tan(4\theta) = \tan\theta\tan(2\theta)\tan(4\theta) = \zeta$$

Figure 66. *Equation 11* Relationship to a Right Triangle when 3θ=60°.



As indicated, three hypotenuses are represented for right triangles comprised of the angles θ, 2θ, and 4θ, respectively. Their interconnections form respective angles of $90^o - 3\theta$, with one another, computed as follows:

$$(90^\circ - \theta)] - 2\theta = 90^\circ - 3\theta$$

$$4\theta - (90^\circ - 2\theta) = 6\theta - 90^\circ = 120^o - 90^o = (180^o - 60^o) - 90^o = (180^o - 3\theta) - 90^o = 90^o - 3\theta$$

The terminations of these additional hypotenuses connect to the ends of the 30°, 60°, right triangle hypotenuse which extends from the origin to point "E' located at (1,ζ).

PROBLEM NUMBER 8 *(Ref. Section 2.4.3)*

GIVEN:

Equation 12 as follows:

$$z_1 + z_2 + z_3 = 3\zeta \qquad \text{[Ref. Equation 12]}$$

Where,

$$z_1 = \tan\theta$$

$$z_2 = \tan\theta + 120^o$$

$$z_3 = \tan\theta + 240^o$$

CONSTRUCT:

A geometry which relates the above given relationship to a right triangle for $3\theta = 60^o$

SOLUTION:

Where

$$3\theta = 60^o$$

$$\theta = 20^o$$

$$z_1 = \tan\theta = \tan 20^o$$

$$\begin{aligned}\theta + 120^o &= 20^o + 120^o \\ &= 140^o \\ &= 180^o - 40^o \\ &= 180^o - 2(20^o) \\ &= 180^o - 2\theta\end{aligned}$$

$$\begin{aligned}z_2 &= \tan(180^o - 2\theta) \\ &= -\tan 2\theta\end{aligned}$$

$$\begin{aligned}\theta + 240^o &= 20^o + 240^o \\ &= 260^o \\ &= 180^o + 80^o \\ &= 180^o + 4(20^o) \\ &= 180^o + 4\theta\end{aligned}$$

$$\begin{aligned}z_3 &= \tan(180^o + 4\theta) \\ &= \tan 4\theta\end{aligned}$$

Then,

$$z_1 + z_2 + z_3 = 3\zeta$$

$$\tan\theta - \tan(2\theta) + \tan(4\theta) = 3\zeta$$

$$\tan\theta + \tan(4\theta) = 3\zeta + \tan(2\theta)$$

Figure 67 portrays *three sections* of x-axis <u>unit</u> spacing, each of which contains its own straight line. Going from left to right, the respective heights of each line are as follows:

$$z_1 = \tan \theta^o$$

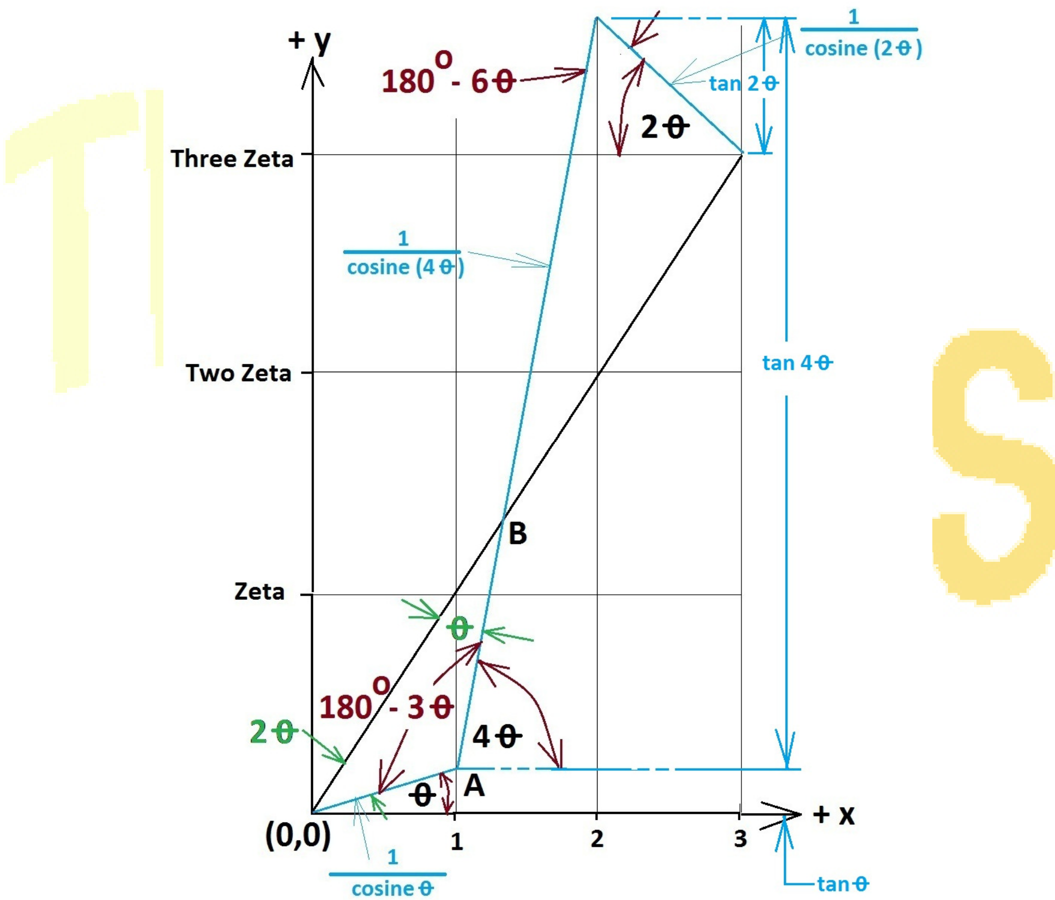
$$z_3 = \tan 4\theta$$

$$z_2 = -\tan 2\theta$$

As indicated,

$$\tan \theta + \tan(4\theta) = 3\zeta + \tan(2\theta)$$

Figure 67. *Equation 12* Relationship to a Right Triangle when 3θ=60°.



The diagonal straight line which connects the origin to point (3, 3ζ) lies at an angle of 3θ with the x-axis, and thereby describes the hypotenuse of a 30°, 60°, right triangle.

As indicated, the $1/\cos\theta$, $1/\cos(4\theta)$ and $1/\cos(2\theta)$ straight lines depict additional hypotenuses for right triangles comprised of the angles θ, 4θ, and -2θ, respectively. Their interconnections form respective angles of $180^o - 3\theta$, and $180^o - 6\theta$ with one another, computed as follows:

$$[180^\circ - (90^\circ - \theta)] + (90^\circ - 4\theta) = 180^\circ - 3\theta$$

$$(90^\circ - 4\theta) + (90^\circ - 2\theta) = 180^\circ - 6\theta$$

The terminations of these additional hypotenuses connect to the ends of the aforementioned 30°, 60°, right triangle hypotenuse.

Lastly, the portion of straight line residing between point A and point B is of length 2, determined as follows

From the Law of Sines:

$$\frac{1/\cos\theta}{\sin\theta} = \frac{\overline{AB}}{\sin(2\theta)}$$

Or,

$$\frac{1/\cos\theta}{\sin\theta} = \frac{\overline{AB}}{2\sin\theta\cos\theta}$$

$$\frac{1}{\cos\theta} = \frac{\overline{AB}}{2\cos\theta}$$

$$1 = \frac{\overline{AB}}{2}$$

$$2 = \overline{AB}$$

PROBLEM NUMBER 9 *(Ref. Section 3.4)*

GIVEN:

A *mole* of *unattached molecules* possessing the *same material, mass*, and *approximate velocity* collide against one another during an experiment where 25.221 x 10^7 foot pounds of *added energy* is applied at time of impact.

Each *molecule* is known to exhibit the following properties:

$$m = 0.422 \text{x} 10^{-21} \text{ slugs}$$

$$\varepsilon_{collision} = 0.08662$$

Twenty percent of the *molecules* are expected to miss each other and, therefore, should not collide.

DETERMINE:

The necessary *initial* and *final velocities*, v_i and v_f, in order to a achieve an *Activation Energy*, 'E_A', of 28.21265 x 10^9 foot-pounds, without considering any added energy gained by secondary collisions.

SOLUTION:

The *activation energy*, E_A, considered necessary to reach threshold represents the summation between the amount of *work* expected to be realized from the initial bombardment of particles and the additional amount of energy afforded, as indicated below:

$$E_A = W + 25.221 \text{ x } 10^7 = 28.21265 \text{ x } 10^9$$

$$W = 28.21265 \text{ x } 10^9 - 25.221(\text{ x } 10^{-2} \text{ x } 10^2) \text{x } 10^7$$

$$= 28.21265 \text{ x } 10^9 - 0.25221 \text{x } 10^9$$

$$= 27.96044 \text{ x } 10^9 \text{ ft-lbs}$$

At 80 percent *collision efficiency*, a *mole of molecules* renders the following amount of collisions:

$$\frac{6.02 \text{ x } 10^{23} \, molecules^{\text{Footnote 1}}}{mole} \, x \, \frac{1 collision}{2 molecules} \, x 0.8 = 2.408 \text{x } 10^{23} \, collisions$$

Hence, the amount of work needed to be realized per collision is calculated as follows:

$$\text{W per collision} = \frac{27.96044 \text{ x } 10^9 \, ft.lbs.}{2.408 \text{x } 10^{23} \, collisions}$$

$$= 11.61147841 \text{ x } 10^{-14} \, ft.lbs$$

1. College Physics Third Edition; Sears & Zemansky; Addison Wesley Publishing Company, Reading, Massachusetts; 1960; page 361.

Since every impact involves molecules of the same initial *material, mass* and *approximate velocity*, the *Complex Quadratic Equation for Work Produced when Two Particles of Constant Mass and Velocity Collide* applies, stipulated as follows:

$$W - m(v_i^2 - v_f^2) = 0 \quad \textit{[Ref. Section 3.4.2]}$$

Where,

$$-\frac{v_f}{v_i} = \varepsilon_{collision} \quad \textit{[Ref. Section 3.4.2]}$$

$$v_f = -(\varepsilon_{\text{collision}})v_i$$
$$= -0.08662 v_i$$

Then, substituting this expression for v_f into the *Constant Input Mass and Velocity Work Relationship* above renders:

$$\text{W per collision} = m[v_1^2 - v_f^2]$$
$$= m[v_1^2 - (-0.08662 v_i)^2]$$
$$= m[v_1^2 - 0.007503 v_i^2]$$
$$= 0.992497 m v_i^2$$
$$= 0.992497(0.422\text{x}10^{-21}\, slugs)v_i^2$$

$$11.61147841\text{ x }10^{-14}\, ft.lbs = (0.418834\text{x}10^{-21}\, slugs)v_i^2$$

$$\frac{11.61147841\text{ x }10^{-14}\, ft.lbs}{0.418834\text{x}10^{-21}\, slugs}\, x\, \frac{1 slug\,(ft/\sec)^2}{1\, ft.lb} = v_i^2$$

$$27.72334245(ft/\sec)^2 = v_i^2$$

$$5.265296045\, ft/\sec\, x\, \frac{1 mile}{5{,}280\, ft}\, x\, \frac{3600 \sec}{hour} = v_i$$

Or approximately,

$$3.6 mph = v_i$$

$$v_f = -0.08662 v_i$$
$$= -0.08662(5.3)$$
$$= -0.46\, ft/\sec\, x\, \frac{12 inches}{ft}$$
$$= -5.5 inches/\sec$$

<u>In conclusion</u>, this reaction results in a final velocity, 'v_f' of approximately one-tenth (0.1) that of the initial velocity, 'v_i', which was imparted to the particles. Hence, a fairly large percentage of the initial energy was converted into work, indicative of the very low $\varepsilon_{collision}$ ratio.

PROBLEM NUMBER 10 *(Ref. Section 10)*

GIVEN:

$$R = S = T = 1$$

SHOW:

Show that the above given set of conditions satisfies the *Unified Cubic Trigonometric Reduction Equation (Ref. Equation 29)*, but cannot be used to resolve it.

SOLUTION:

When $R = S = T = 1$:

$$z_R = R\tan\theta = 1\tan\theta = \tan\theta = \tan\theta_R$$
$$z_S = S\tan\theta = 1\tan\theta = \tan\theta = \tan\theta_S$$
$$z_T = T\tan\theta = 1\tan\theta = \tan\theta = \tan\theta_T$$

Equation 29 is satisfied whenever:

$$\theta_R + \theta_S + \theta_T = 3\theta = \theta + \theta + \theta$$

Such that,

$$R + S + T = 1 + 1 + 1 = 3$$
$$RS + RT + ST = 1 + 1 + 1 = 3$$
$$RST = 1(1)(1) = 1$$

For *Equation 29*:

$$\zeta(RST - 1) + [(R + S + T) - 3RST]\tan\theta + \zeta[(RS + RT + ST) - 3RST]\tan^2\theta = 0$$ *[Ref. Equation 29]*

$$\zeta(1 - 1) + [3 - 3(1)]\tan\theta + \zeta[(3) - 3(1)]\tan^2\theta = 0$$
$$0 + 0\tan\theta + 0\tan^2\theta = 0$$

Or,

$$0 + 0 + 0 = 0$$

This results in an *identity* which cannot contribute to the resolution of *Equation 29*.

PROBLEM NUMBER 11 *(Ref. Section 10)*

GIVEN:

The Unified Cubic Trigonometric Reduction Equation (Ref. Equation 29)

DETERMINE:

a) A *formula* which renders "T" in terms of ζ, tan θ, R and S

b) The value for "T" when $R = 2$, $S = 1$, and $\zeta = \tan(3\theta) = \tan 60^o = \sqrt{3}$ using the formula established above

SOLUTION:

Table 37 For *Equation 29:*

$$\zeta(RST - 1) + [(R + S + T) - 3RST]\tan\theta + \zeta[(RS + RT + ST) - 3RST]\tan^2\theta = 0 \qquad \textit{[Ref. Equation 29]}$$

$$T[\zeta RS + (1 - 3RS)\tan\theta + \zeta(R + S - 3RS)\tan^2\theta] = \zeta - (R + S)\tan\theta - \zeta(RS)\tan^2\theta$$

Or,

$$T = \frac{\zeta - (R + S)\tan\theta - \zeta(RS)\tan^2\theta}{\zeta RS + (1 - 3RS)\tan\theta + \zeta(R + S - 3RS)\tan^2\theta}$$

Table 38 Where,

$$\theta = \frac{\tan 60^o}{3} = 20^o$$

$$\tan 20^o = 0.363970234$$

$$\begin{aligned}
T &= \frac{\zeta - (R + S)\tan\theta - \zeta(RS)\tan^2\theta}{\zeta RS + (1 - 3RS)\tan\theta + \zeta(R + S - 3RS)\tan^2\theta} \\
&= \frac{\sqrt{3} - (2 + 1)(0.363970234) - \sqrt{3}(2)(1)(0.363970234)^2}{\sqrt{3}(2)(1) + [1 - 3(2)(1)](0.363970234) + \sqrt{3}[2 + 1 - 3(2)(1)](0.363970234)^2} \\
&= \frac{\sqrt{3} - (3)(0.363970234) - 2\sqrt{3}(0.363970234)^2}{2\sqrt{3} - 5(0.363970234) - \sqrt{3}(3)(0.363970234)^2} \\
&= \frac{\sqrt{3}[1 - \sqrt{3}(0.363970234) - 2(0.363970234)^2]}{\sqrt{3}[2 - \frac{5}{\sqrt{3}}(0.363970234) - (3)(0.363970234)^2]} \\
&= \frac{1 - \sqrt{3}(0.363970234) - 2(0.363970234)^2}{2 - \frac{5}{\sqrt{3}}(0.363970234) - (3)(0.363970234)^2} \\
&= \frac{0.104636398}{0.551885442} \\
&= 0.18959804
\end{aligned}$$

PROBLEM NUMBER 12 *(Ref. Section 10)*

GIVEN:

The equation:

$$2.478921295x^2 + 2.052009795\,x - 1.075263928 = 0$$

DETERMINE:

"S" and "T" for the particular condition when $R = 2$

SOLUTION:

For the *given Quadratic Equation*:

$$ax^2 + b\,x + c = 0$$

$$2.478921295x^2 + 2.052009795\,x - 1.075263928 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2.052009795 \pm \sqrt{(2.052009795)^2 + 4(2.478921295)(1.075263928)}}{2(2.478921295)}$$

$$= \frac{-2.052009795 \pm \sqrt{4.21074199 + 10.6619786}}{2(2.478921295)}$$

$$= \frac{-2.052009795 \pm \sqrt{14.87272279}}{2(2.478921295)}$$

$$= \frac{-2.052009795 \pm 3.856516925}{2(2.478921295)}$$

$$= 0.363970234; -1.191753593$$

$$= \tan 20^o; -\frac{1}{\tan 40^o}$$

$$\zeta = \tan(3\theta) = \tan 60^o = \sqrt{3}$$

Comparison between the coefficients for each of the terms expressed in the *Unified Cubic Trigonometric Reduction Equation (Ref. Equation 29)* shown below with those indicated in the *Quadratic Equation* renders the following set of relationships*:*

[Ref. Equation 29] $$\zeta(RST - 1) + [(R + S + T) - 3RST]\tan\theta + \zeta[(RS + RT + ST) - 3RST]\tan^2\theta = 0$$

$$\zeta[(RS + RT + ST) - 3RST]\tan^2\theta + [(R + S + T) - 3RST]\tan\theta + \zeta(RST - 1) = 0$$

$$ax^2 + b\,x + c = 0$$

$$a = \zeta[(RS + RT + ST) - 3RST]$$

$$b = R + S + T - 3RST$$

$$c = \zeta(RST - 1)$$

Then,

$$RST-1=\frac{c}{\zeta}$$

$$RST=\frac{c+\zeta}{\zeta}$$

$$a=\zeta[(RS+RT+ST)-3RST]$$

$$=\zeta[(RS+RT+ST)-3(\frac{c+\zeta}{\zeta})]$$

$$=\sqrt{3}[(RS+RT+ST)-3(\frac{c+\sqrt{3}}{\sqrt{3}})]$$

$$b=R+S+T-3RST$$

$$=R+S+T-3(\frac{c+\zeta}{\zeta})$$

$$=2+S+T-3(\frac{c+\sqrt{3}}{\sqrt{3}})$$

$$=2.052009795$$

Then,

$$2.052009795=2+S+T-3(\frac{c+\sqrt{3}}{\sqrt{3}})$$

$$0.052009795+3(\frac{c+\sqrt{3}}{\sqrt{3}})-T=S$$

$$0.052009795+3(\frac{-1.075263928+\sqrt{3}}{\sqrt{3}})-T=S$$

$$0.052009795+3(0.379196081)-T=S$$

$$0.052009795+1.137588245-T=S$$

$$1.189598041-T=S$$

$$1.189598041=S+T$$

But,

$$a=2.478921295=\sqrt{3}[(RS+RT+ST)-3(\frac{c+\sqrt{3}}{\sqrt{3}})]$$

$$=\sqrt{3}[2(S+T)+ST-3(\frac{c+\sqrt{3}}{\sqrt{3}})]$$

$$=\sqrt{3}[2(1.189598041)+(1.189598041-T)T-3(\frac{c+\sqrt{3}}{\sqrt{3}})]$$

$$=\sqrt{3}[2.379196082+(1.189598041-T)T-1.137588245]$$

$$2.478921295+\sqrt{3}(1.137588245)=\sqrt{3}(2.379196082+1.189598041T-T^2)$$

$$2.478921295 + 1.97036064 = \sqrt{3}(2.379196082 + 1.189598041T - T^2)$$

$$4.449281935 = \sqrt{3}(2.379196082 + 1.189598041T - T^2)$$

$$\frac{4.449281935}{\sqrt{3}} = 2.379196082 + 1.189598041T - T^2$$

$$2.568794123 = 2.379196082 + 1.189598041T - T^2$$

$$2.568794123 - 2.379196082 = 1.189598041T - T^2$$

$$T^2 - 1.189598041T + 0.189598041 = 0$$

Then,

$$S;T = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1.189598041 \pm \sqrt{(1.189598041)^2 - 4(1)(0.189598041)}}{2}$$

$$= \frac{1.189598041 \pm \sqrt{0.656751335}}{2}$$

$$= \frac{1.189598041 \pm 0.810401959}{2}$$

$$= 1; 0.189598041$$

PROBLEM NUMBER 13 *(Ref. Section 10)*

GIVEN:

The following equation:

$$2.478921295x^2 + 2.052009795\,x - 1.075263928 = 0$$

PROVE:

That a value of $R = 3$ cannot apply to the above *given equation*

SOLUTION:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2.052009805 \pm \sqrt{(2.052009805)^2 + 4(2.478921305)(1.075263932)}}{2(2.478921305)}$$

$$= 0.363970234; -1.19753593$$

$$= \tan 20^o; -\frac{1}{\tan 40^o}$$

$$\zeta = \tan 3\theta = \tan 60^o = \sqrt{3}$$

From *Equation 29,*

$$a = \zeta[(RS + RT + ST) - 3RST]$$

$$b = R + S + T - 3RST$$

$$c = \zeta(RST - 1)$$

Where,

$$ax^2 + b\,x + c = 0$$

Then,

$$\frac{c + \zeta}{\zeta} = RST$$

$$b = 2.052009795 = R + S + T - 3RST$$

$$= R + S + T - 3(\frac{c + \sqrt{3}}{\sqrt{3}})$$

$$= 3 + S + T - 3(\frac{c + \sqrt{3}}{\sqrt{3}})$$

$$-0.947990205 + 3(\frac{c + \sqrt{3}}{\sqrt{3}}) - T = S$$

$$-0.947990205 + 3(\frac{-1.075263928 + \sqrt{3}}{\sqrt{3}}) - T = S$$

$$-0.947990195 + 1.137588245 - T = S$$

$$0189598041 - T = S$$

$$a = 2.478921295 = \zeta[(RS + RT + ST) - 3RST]$$
$$= \sqrt{3}[3S + 3T + ST - 3(\frac{c+\sqrt{3}}{\sqrt{3}})]$$
$$= \sqrt{3}[3(0.189598041 - T) + 3T + (0.189598041 - T)T - 3(\frac{-1.075263928 + \sqrt{3}}{\sqrt{3}})]$$
$$= \sqrt{3}[3(0.189598041) + (0.189598041 - T)T - 3(0.379196081)]$$
$$= \sqrt{3}[3(0.189598041) + (0.189598041 - T)T - 1.137588245]$$
$$= \sqrt{3}[-0.568794122 + (0.189598043 - T)T]$$

Or,

$$2.478921295 = \sqrt{3}[-0.568794122 + 0.189598041T - T^2]$$
$$1.431205877 = -0.568794122 + 0.189598041T - T^2$$
$$2 = 0.189598041T - T^2$$

Then,

$$T^2 - 0.189598041T + 2 = 0$$
$$S;T = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{0.189598041 \pm \sqrt{(0.189598041)^2 - 4(1)(2)}}{2}$$
$$= \frac{0.189598041 \pm \sqrt{-7.964052583}}{2}$$
$$= \frac{0.189598041 \pm 2.822065305i}{2}$$

As indicated in the last three columns of the table below, *mathematical expressions* pertaining to the ***real RST Spread*** don't always match up to respective ones pertaining to the *RST Spread* calculated above. This is because only one *RST Spread*, or interchangeable grouping, can satisfy any equation such as the one given below.

$$ax^2 + bx + c = 0$$
$$2.478921295x^2 + 2.052009795x - 1.075263928 = 0$$

	$R+S+T$	$RS+RT+ST$	RST
Real RST Spread			
$R=1$, $S=2$, $T=0.189598041$	3.189598041	2.568794123	0.379196082
Calculated RST Spread			
$R=3$, $S=\dfrac{0.189598041+2.822065305i}{2}$, $T=\dfrac{0.189598041-2.822065305i}{2}$	3.189598041	−1.413232168	−5.946078874

A determination of the *coefficients* for the above *given equation* using *RST Spread calculated values* is rendered below:

$$\begin{aligned} a &= \zeta[(RS+RT+ST)-3RST] \\ &= \sqrt{3}[-1.413232168-3(-5.946078874)] \\ &= \sqrt{3}(-1.413232168+17.83823662) \\ &= \sqrt{3}(16.42500445) \\ &= 28.444894223 \neq 2.478921295 \end{aligned}$$

$$\begin{aligned} b &= R+S+T-3RST \\ &= 3.189598041+17.83823662 \\ &= 21.02783466 \neq 2.052009795 \end{aligned}$$

$$\begin{aligned} c &= \zeta(RST-1) \\ &= \sqrt{3}(-5.946078874-1) \\ &= \sqrt{3}(-6.946078874) \\ &= -12.03096152 \neq -1.075263928 \end{aligned}$$

Since the calculated coefficients don't match those for the above *given equation*, the *RST Spread* determined above must be incorrect, or <u>cannot</u> exist.

PROBLEM NUMBER 14 *(Ref. Section 10)*

GIVEN:

The following equation:

$$2.478921295x^2 + 2.052009795\,x - 1.075263928 = 0$$
$$ax^2 + b\,x + c = 0$$

PROVE:

That a particular value of $R = 3/4$ cannot apply to the above *given equation*

SOLUTION:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2.052009805 \pm \sqrt{(2.052009805)^2 + 4(2.478921305)(1.075263932)}}{2(2.478921305)}$$

$$= 0.363970234; -1.19753593$$

$$= \tan 20^o; -\frac{1}{\tan 40^o}$$

$$\zeta = \tan 3\theta = \tan 60^o = \sqrt{3}$$

From *Equation 29*,

$$a = \zeta[(RS + RT + ST) - 3RST]$$

$$b = R + S + T - 3RST$$

$$c = \zeta(RST - 1)$$

Where,

$$ax^2 + b\,x + c = 0$$

Then,

$$c = \zeta(RST - 1)$$

$$\frac{c + \zeta}{\zeta} = RST$$

$$b = 2.052009795 = R + S + T - 3RST$$

$$= R + S + T - 3(\frac{c + \sqrt{3}}{\sqrt{3}})$$

$$= \frac{3}{4} + S + T - 3(\frac{c + \sqrt{3}}{\sqrt{3}})$$

$$1.302009795 + 3(\frac{c + \sqrt{3}}{\sqrt{3}}) - T = S$$

$$1.302009795 + 3(\frac{-1.075263928 + \sqrt{3}}{\sqrt{3}}) - T = S$$
$$1.302009795 + 3(0.379196081) - T = S$$
$$1.302009795 + 1.137588245 - T = S$$
$$2.43959804 - T = S$$

$$a = 2.478921295 = \zeta[(RS + RT + ST) - 3RST]$$
$$= \sqrt{3}[\frac{3}{4}S + \frac{3}{4}T + ST - 3(\frac{c + \sqrt{3}}{\sqrt{3}})]$$
$$= \sqrt{3}[\frac{3}{4}(2.43959804 - T) + \frac{3}{4}T + (2.43959804 - T)T - 3(\frac{-1.075263928 + \sqrt{3}}{\sqrt{3}})]$$
$$= \sqrt{3}[\frac{3}{4}(2.43959804) + (2.43959804 - T)T - 3(\frac{-1.075263928 + \sqrt{3}}{\sqrt{3}})]$$
$$= \sqrt{3}[\frac{3}{4}(2.43959804) + (2.43959804 - T)T - 3(0.379196081)]$$
$$= \sqrt{3}[1.82969853 + (2.43959804 - T)T - 1.137588245]$$
$$= \sqrt{3}[0.692110285 + (2.43959804 - T)T]$$

Or,

$$2.478921295 = \sqrt{3}[0.692110285 + (2.43959804 - T)T]$$
$$1.431205877 = 0.692110285 + (2.43959804 - T)T$$
$$0.739095591 = 2.43959804T - T^2$$

Then,

$$T^2 - 2.43959804T + 0.739095591 = 0$$
$$S;T = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{2.43959804 \pm \sqrt{(-2.43959804)^2 - 4(1)(0.739095591)}}{2}$$
$$= \frac{2.43959804 \pm \sqrt{5.951638597 - 2.956382366}}{2}$$
$$= \frac{2.43959804 \pm \sqrt{2.99525623}}{2}$$
$$= \frac{2.43959804 \pm 1.730680857}{2}$$
$$= 2.085139449; 0.354458591$$

As indicated in the last three columns of the table below, *mathematical expressions* pertaining to the ***real RST Spread*** don't always match up to respective ones pertaining to the *RST Spread* calculated above. This is because only one *RST*

Spread, or interchangeable grouping, <u>can</u> satisfy any equation such as the one given below.

$$ax^2 + bx + c = 0$$

$$2.478921295x^2 + 2.052009795x - 1.075263928 = 0$$

	$R+S+T$	$RS+RT+ST$	RST
<u>Real RST Spread</u>			
$R = 1$ $S = 2$ $T = 0.189598041$	3.189598041	2.568794123	0.379196082
<u>Calculated RST Spread</u>			
$R = 0.75$ $S = 2.085139449$ $T = 0.354458591$	3.189598041	2.568794123	0.554321693

A determination of the *coefficients* for the above *given equation* using *RST Spread calculated values* is rendered below:

$$\begin{aligned} a &= \zeta[(RS + RT + ST) - 3RST] \\ &= \sqrt{3}[2.568794123 - 3(0.554321693)] \\ &= \sqrt{3}(2.568794123 - 1.662965079) \\ &= \sqrt{3}(0.905829044) \\ &= 1.568941927 \neq 2.478921295 \end{aligned}$$

$$\begin{aligned} b &= R + S + T - 3RST \\ &= 3.189598041 - 1.662965079 \\ &= 1.526632962 \neq 2.052009795 \end{aligned}$$

$$\begin{aligned} c &= \zeta(RST - 1) \\ &= \sqrt{3}(0.554321693 - 1) \\ &= \sqrt{3}(-0.445678307) \\ &= -0.771937471 \neq -1.075263928 \end{aligned}$$

Since the calculated coefficients don't match those for the above *given equation*, the *RST Spread* determined above must be incorrect, or <u>cannot</u> exist.

PROBLEM NUMBER 15 *(Ref. Section 11.1)*

GIVEN:

The *Quadratic Equation* $ax^2 + bx + c = 0$

DETERMINE:

a) A general formula in terms of ζ and T for the specific condition when $R=1$ and $S=2$

b) The specific equation for $T = \frac{1}{32}$

SOLUTION:

Table 39 For,

$$ax^2 + bx + c = 0$$

$$R=1$$
$$S=2$$

Where,

$$\zeta(C+3D)\tan^2\theta - (B-3D)\tan\theta - \zeta(D+1) = 0$$ *[Ref. Equation 30]*

Being of the form:

$$a\tan^2\theta + b\tan\theta + c = 0$$

And,

$$B = -(R+S+T)$$
$$C = RS + RT + ST$$
$$D = -RST$$

Comparing like coefficients renders,

$$\begin{aligned} a = \zeta(C+3D) &= \zeta[(RS+RT+ST) - 3RST] \\ &= \zeta[RS + (R+S)T - 3RST] \\ &= \zeta[(1)(2) + (1+2)T - 3(1)(2)T] \\ &= \zeta[(2+3T) - 6T] \\ &= \zeta(2-3T) \end{aligned}$$

$$\begin{aligned} b = -(B-3D) &= R+S+T-3RST \\ &= (1+2+T) - 3(1)(2)T \\ &= 3-5T \end{aligned}$$

$$\begin{aligned} c = \zeta(D+1) &= \zeta(RST-1) \\ &= \zeta[(1)(2)T-1] \\ &= \zeta(2T-1) \end{aligned}$$

Then,

$$\zeta(2-3T)x^2 + (3-5T)x + \zeta(2T-1) = 0$$

b)For,

$$\zeta(2-3T)x^2+(3-5T)x+\zeta(2T-1)=0$$

$$\zeta[2-3(\frac{1}{32})]x^2+[3-5(\frac{1}{32})]x+\zeta[2(\frac{1}{32})-1]=0$$

$$\zeta(\frac{64-3}{32})x^2+(\frac{96-5}{32})x+\zeta(\frac{2-32}{32})=0$$

$$\zeta(\frac{61}{32})x^2+(\frac{91}{32})x-\zeta(\frac{30}{32})=0$$

$$61\zeta x^2+91x-30\zeta=0$$

The value of $\zeta=\tan(3\theta)$ is determined as follows:

$$z_R=R\tan\theta=1\tan\theta=\tan\theta=\tan\theta_R$$

$$z_S=S\tan\theta=2\tan\theta=\tan\theta_S$$

$$z_T=T\tan\theta=\frac{1}{32}\tan\theta=\tan\theta_T$$

Where,

$$\theta=\theta_R$$

$$3\theta=\theta_R+\theta_S+\theta_T$$

$$=\theta+\theta_S+\theta_T$$

$$2\theta=\theta_S+\theta_T$$

$$\tan(2\theta)=\tan(\theta_S+\theta_T)$$

$$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{\tan\theta_S+\tan\theta_T}{1-\tan\theta_S\tan\theta_T}$$

$$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2\tan\theta+\frac{1}{32}\tan\theta}{1-\frac{1}{16}\tan^2\theta}$$

$$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2\tan\theta(1+\frac{1}{64})}{1-\frac{1}{16}\tan^2\theta}$$

$$\frac{1}{1-\tan^2\theta}=\frac{1+\frac{1}{64}}{1-\frac{1}{16}\tan^2\theta}$$

$$\frac{1}{1-\tan^2\theta}=\frac{65}{64-4\tan^2\theta}$$

$$64-4\tan^2\theta=65(1-\tan^2\theta)$$

$$61\tan^2\theta=1$$

$$\tan^2\theta = \frac{1}{61}$$

$$\tan\theta = 0.128035879$$

$$\theta = 7.296275301^o$$

$$3\theta = 21.8888259^o$$

$$\zeta = \tan(3\theta) = 0.401770899$$

Via substitution,

$$61\zeta x^2 + 91x - 30\zeta = 0$$

$$61(0.401770899)x^2 + 91x - 30(0.401770899) = 0$$

$$2.450802484x^2 + 91x - 12.05312694 = 0$$

Check,

$$\tan\theta_R = R\tan\theta = 1\tan\theta = \tan\theta$$

$$\tan\theta_S = S\tan\theta = 2\tan\theta = 2(0.128035879) = 0.256071758$$

$$\tan\theta_T = T\tan\theta = \frac{1}{32}\tan\theta = \frac{1}{32}(0.128035879) = 0.004001121$$

$$\theta_R = \theta = 7.296275301^o$$

$$\theta_S = \arctan 0.256071758 = 14.36319503^o$$

$$\theta_T = \arctan 0.004001121 = \underline{0.229246135^o}$$

$$\Sigma = 21.8888^o$$

$$= 3\theta$$

Then,

$$\zeta(2-3T)x^2 + (3-5T)x + \zeta(2T-1) = 0$$

$$0.401770899[2-3(\frac{1}{32})]x^2 + [3-5(\frac{1}{32})]x + 0.401770899[2(\frac{1}{32})-1] = 0$$

$$0.765875776x^2 + 2.84375x - 0.376660217 = 0$$

$$0.765875776(32)x^2 + 2.84375(32)x - 0.376660217(32) = 0$$

$$2.450802484x^2 + 91x - 12.05312694 = 0$$ *Q.E.D.*

PROBLEM NUMBER 16 *(Ref. Section 11.1)*

GIVEN:

The reduced *Quadratic Equation* $x^2 + bx + c = 0$

FIND:

The associated ***RST Spread*** when $\zeta = \sqrt{3}$

SOLUTION:

Since real values of tan θ exist from minus infinity to plus infinity, it may be used to represent x, such that:

$$x = \tan\theta$$

Via substitution, the above *given reduced Quadratic Equation* may be expressed as:

$$\tan^2\theta + b\tan\theta + c = 0$$

The associated ***RST Spread*** is determined by comparing like coefficients between the above *given Quadratic Equation* and the *Simplified Unified Cubic Trigonometric Reduction Equation (Ref. Equation 30)* as follows:

Where,

$$\zeta(C+3D)\tan^2\theta - (B-3D)\tan\theta - \zeta(D+1) = 0 \quad \textit{[Ref. Equation 30]}$$

Being of the form:

$$a\tan^2\theta + b\tan\theta + c = 0$$

And,

$$B = -(R+S+T)$$
$$C = RS + RT + ST$$
$$D = -RST$$

Comparing like coefficients renders,

$$a = \zeta(C+3D) = \zeta[(RS+RT+ST) - 3RST]$$

$$b = -(B-3D) = R+S+T-3RST$$

$$c = \zeta(D+1) = \zeta(RST-1)\,,\ \backslash$$

Then by further comparison of coefficients,

$$a = \zeta[(RS+RT+ST) - 3RST] = 1$$

$$RS + RT + ST - 3RST = \frac{1}{\zeta}$$

Whereby the *first root* for the above *given reduced Quadratic Equation* is tan θ, then the value of R, S, or T must be equal to unity.

As such,

$$RS + RT + ST - 3RST = \frac{1}{\zeta}$$

$$S + T + ST - 3ST = \frac{1}{\zeta}$$

$$S + T - 2ST = \frac{1}{\zeta}$$

$$S(1-2T) + \frac{\zeta T}{\zeta} = \frac{1}{\zeta}$$

$$S = \frac{1-\zeta T}{\zeta(1-2T)}$$

Now, considering that

$$z_R = R\tan\theta = (1)\tan\theta = \tan\theta = \tan\theta_R$$

$$\theta = \theta_R$$

Where *(Ref. Section 10)*,

$$3\theta = \theta_R + \theta_S + \theta_T$$

$$3\theta = \theta + \theta_S + \theta_T$$

$$2\theta = \theta_S + \theta_T$$

$$2\theta - \theta_S = \theta_T$$

Whereby,

$$\tan\theta_T = T\tan\theta = \tan(2\theta - \theta_S)$$

$$= \frac{\tan(2\theta) - \tan\theta_S}{1 + \tan(2\theta)\tan\theta_S}$$

$$= \frac{\tan(2\theta) - S\tan\theta}{1 + S\tan\theta\tan(2\theta)}$$

$$= \frac{2\tan\theta - S\tan\theta(1-\tan^2\theta)}{1 - \tan^2\theta + 2S\tan^2\theta}$$

$$= \frac{(2-S)\tan\theta + S\tan^3\theta}{1 + (2S-1)\tan^2\theta}$$

Substituting for the value of S above gives,

$$2 - S = 2 - \frac{1-\zeta T}{\zeta(1-2T)}$$

$$= \frac{2\zeta(1-2T) - (1-\zeta T)}{\zeta(1-2T)}$$

$$= \frac{(2\zeta - 1) - 3\zeta T}{\zeta(1-2T)}$$

And,

$$2S-1=\frac{2(1-\zeta T)-\zeta(1-2T)}{\zeta(1-2T)}$$
$$=\frac{2-\zeta}{\zeta(1-2T)}$$

By again substituting for $2-S$, S, and $2S-1$ above, the following expression is obtained:

$$T\tan\theta=\frac{(2-S)\tan\theta+S\tan^3\theta}{1+(2S-1)\tan^2\theta}$$
$$=\frac{\frac{(2\zeta-1)-3\zeta T}{\zeta(1-2T)}\tan\theta+\frac{1-\zeta T}{\zeta(1-2T)}\tan^3\theta}{1+\frac{2-\zeta}{\zeta(1-2T)}\tan^2\theta}$$

Or,

$$T=\frac{\frac{(2\zeta-1)-3\zeta T}{\zeta(1-2T)}+\frac{1-\zeta T}{\zeta(1-2T)}\tan^2\theta}{1+\frac{2-\zeta}{\zeta(1-2T)}\tan^2\theta}$$
$$=\frac{(2\zeta-1)-3\zeta T+(1-\zeta T)\tan^2\theta}{\zeta(1-2T)+(2-\zeta)\tan^2\theta}$$
$$T[\zeta(1-2T)+(2-\zeta)\tan^2\theta]=2\zeta-1-3\zeta T+(1-\zeta T)\tan^2\theta$$
$$\zeta T-2\zeta T^2+(2-\zeta)T\tan^2\theta=2\zeta-1-3\zeta T+(1-\zeta T)\tan^2\theta$$
$$\zeta T-2\zeta T^2+2T\tan^2\theta=2\zeta-1-3\zeta T+\tan^2\theta$$
$$0=2\zeta T^2-(3\zeta+\zeta+2\tan^2\theta)T+(2\zeta-1+\tan^2\theta)$$
$$0=2\zeta T^2-(4\zeta+2\tan^2\theta)T+(2\zeta-1+\tan^2\theta)$$
$$0=T^2-(\frac{4\zeta+2\tan^2\theta}{2\zeta})T+\frac{2\zeta-1+\tan^2\theta}{2\zeta}$$

For

$$\zeta=\sqrt{3}=\tan(3\theta)=\tan 60^o$$
$$3\theta=60^o$$
$$\theta=20^o$$

$$\tan\theta=\tan 20^o=0.363970234$$

$$0=T^2-[\frac{4\sqrt{3}+2(0.363970234)^2}{2\sqrt{3}}]T+\frac{2\sqrt{3}-1+(0.363970234)^2}{2\sqrt{3}}$$
$$0=T^2-2.076484091T+0.74956691$$

$$S,T = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$$

$$= \frac{1}{2(1)}[2.076484091 \pm \sqrt{(-2.076484091)^2 - 4(1)(0.74956691)}]$$

$$= \frac{1}{2}[2.076484091 \pm 1.146088363]$$

$$= 1.611286227; 0.465197863$$

Check:

$$z_R = R\tan\theta = 1\tan\theta = 0.36970234 = \tan 20^o = \tan\theta_R$$

$$z_S = S\tan\theta = 1.611286227\tan\theta = 0.586460225 = \tan 30.38992734^o = \tan\theta_S$$

$$z_T = T\tan\theta = 0.465197863\tan\theta = 0.169318175 = \tan 9.610072646^o = \tan\theta_T$$

$$20^o = \theta_R$$

$$30.38992734^o = \theta_S$$

$$9.610072646^o = \theta_T$$

$$3\theta = \theta_R + \theta_S + \theta_T$$

$$= 20^o + 30.38992734^o + 9.610072646^o$$

$$= 60^o$$

Where,

$$a = \zeta[(RS + RT + ST) - 3RST]$$

$$= \zeta(S + T - 2ST)$$

$$= \zeta[1.611286227 + 0.465197863 - 2(1.611286227)(0.465197863)]$$

$$= \zeta[2.07648409 - 2(0.74956691)]$$

$$= \sqrt{3}(0.577350271)$$

$$= 1$$

PROBLEM NUMBER 17 *(Ref. Section 11.1)*

GIVEN:

The reduced *Quadratic Equation* shown below and the *Known Cubic Equation for the Tangent (3θ)* when $\zeta = \sqrt{3}$ *(Ref. Equation 3)*:

$$2.478921295\tan^2\theta + 2.052009795\tan\theta - 1.075263928 = 0$$

DEMONSTRATE:

How the two above given equations are associated with each other.

SOLUTION:

The roots for the above *given Quadratic Equation* are determined via *Quadratic Formula* as follows:

Where,

$$2.478921295\tan^2\theta + 2.052009795\tan\theta - 1.075263928 = 0$$

Being of the form:

$$a\tan^2\theta + b\tan\theta + c = 0$$

Is solved as follows:

$$\tan\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(2.052009795) \pm \sqrt{(2.052009795)^2 - 4(2.478921295)(-1.075263928)}}{2(2.478921295)}$$

$$= \frac{-(2.052009795) \pm \sqrt{14.87272279}}{2(2.478921295)}$$

$$= \frac{-(2.052009795) \pm 3.856516925}{4.95784259}$$

$$= 0.363970234; -1.19175393$$

Secondly, the *Known Cubic Equation for the Tangent (3θ) (Ref. Equation 3)* is *transformed* into a *reduced Quadratic Equation* for purposes of comparing it to the above results.

This is accomplished by applying the *Simplified Unified Cubic Trigonometric Reduction Equation (Ref. Equation 30)* as follows:

Where,

$$\zeta(C + 3D)\tan^2\theta - (B - 3D)\tan\theta - \zeta(D + 1) = 0 \qquad \text{[Ref. Equation 30]}$$

Being of the form:

$$a\tan^2\theta + b\tan\theta + c = 0$$

And,

$$B = -(R + S + T)$$
$$C = RS + RT + ST$$
$$D = -RST$$

Comparing like coefficients renders,

$$a = \zeta(C + 3D) = \zeta[(RS + RT + ST) - 3RST]$$
$$b = -(B - 3D) = R + S + T - 3RST$$
$$c = \zeta(D + 1) = \zeta(RST - 1)$$

With respect to the *Known Cubic Equation for the Tangent (3θ) (Ref. Equation 3)*, the values of R, S, and T are determined as follows:

$$\tan^3\theta = 3\tan\theta - \zeta(1 - 3\tan^2\theta) \qquad \textit{[Ref. Equation 3]}$$

For

$$\zeta = \sqrt{3} = \tan(3\theta) = \tan 60^o$$
$$3\theta = 60^o$$
$$\theta = 20^o$$

$$\tan\theta = \tan 20^o = 0.363970234$$

Where *(Ref. Section 2.3)*,

$$z_R = R\tan\theta = (1)\tan 20^o$$
$$z_S = S\tan\theta = \tan(\theta + 120^o) = \tan(20^o + 120^o) = \tan 140^o$$
$$z_T = T\tan\theta = \tan(\theta + 240^o) = \tan(20^o + 240^o) = \tan 260^o$$

$$R = \frac{z_R}{\tan 20^o} = \frac{(1)\tan 20^o}{\tan 20^o} = 1$$
$$S = \frac{z_S}{\tan 20^o} = \frac{\tan 140^o}{\tan 20^o} = \frac{-0.839099631}{0.363970234} = -2.305407289$$
$$T = \frac{z_T}{\tan 20^o} = \frac{\tan 260^o}{\tan 20^o} = \frac{5.67128182}{0.363970234} = 15.58171874$$

$$3RST = 3(1)(-2.305407289)(15.58171874) = -107.7666239$$
$$R + S + T = 1 - 2.305407289 + 15.58171874 = 14.27631145$$
$$RS + RT + ST = 1(-2.305407289 + 15058171874) + (-2.305407289)(15.58171874) = -22.64589651$$

Then,

$$a = \zeta[(RS + RT + ST) - 3RST] = \sqrt{3}(-22.64589651 + 107.7666239) = 147.4334246$$
$$b = R + S + T - 3RST = 14.27631145 + 107.7666239 = 122.0429354$$

$$c = \zeta(RST - 1) = \sqrt{3}(\frac{-107.7666239 - 3}{3}) = -63.95114013$$

So, the resulting associated *reduced Quadratic Equation* is:

$$147.4334246\tan^2\theta + 122.0429354\tan\theta - 63.95114013 = 0$$

Now, solving this via *Quadratic Formula* yields:

$$\tan\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-122.0429354 \pm \sqrt{[122.0429354]^2 - 4(147.4334246)(-63.95114013)}}{2(147.4334246)}$$

$$= \frac{-122.0429354 \pm 229.3656901}{294.8668492}$$

$$= 0.36397023; -1.19175359$$

Since the associated *Simplified Unified Cubic Trigonometric Reduction Equation (Ref. Equation 30)* for the *Known Cubic Equation for the Tangent (3θ)* when $\zeta = \sqrt{3}$ *(Ref. Equation 3)* expresses the *same root set* as determined for the *given Quadratic Equation*, then they should be equivalent.

This is verified as follows with respect to the two resulting equations shown below:

$$147.4334246\tan^2\theta + 122.0429354\tan\theta - 63.95114013 = 0$$

$$2.478921295\tan^2\theta + 2.052009795\tan\theta - 1.075263928 = 0$$

Where,

$$\frac{147.4334246}{2.478921295} = 59.47483081$$

Dividing respective terms of the top equation by a factor of 59.47483081 gives:

$$\frac{147.4334246}{59.47483081}\tan^2\theta + \frac{122.0429354}{59.47483081}\tan\theta - \frac{63.95114013}{59.47483081} = \frac{0}{59.47483081}$$

$$2.478921295\tan^2\theta + (2.052009795)\tan\theta - 1.075263928 = 0$$

Since this result represents the *given Quadratic Equation*, the associated *reduced equation*, reiterated below, must be a simple multiple of it.

$$147.4334246\tan^2\theta + 122.0429354\tan\theta - 63.95114013 = 0$$

Hence, the two equations are *equivalent*.

PROBLEM NUMBER 18 *(Ref. Sections 11.1 and 11.3)*

GIVEN:

The *Known Cubic Equation for the Tangent (3θ) (Ref. Equation 3)*, the *Simplified Unified Cubic Trigonometric Reduction Equation (Ref. Equation 30)*, and the *Generalized Cubic Equation (Ref. Equation 32)* shown below:

$$\tan^3\theta = 3\tan\theta - \tan(3\theta)(1-3\tan^2\theta) \qquad \text{[Ref. Equation 3]}$$

$$\zeta(C+3D)\tan^2\theta - (B-3D)\tan\theta - \zeta(D+1) = 0 \qquad \text{[Ref. Equation 30]}$$

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \text{[Ref. Equation 32]}$$

DETERMINE:

A ***general expression*** for the *Simplified Unified Cubic Trigonometric Reduction Equation* in terms of *Generalized Cubic Equation* coefficients

VERIFY:

That **r**oots for such ***general expression*** are $\tan\theta$ and $-1/\tan(2\theta)$ when the *coefficients* for the *3θ Cubic Equation* are substituted for those expressed in the *Generalized Cubic Equation*

SOLUTION:

Determination:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \text{[Ref. Equation 32]}$$

Where,

$$\alpha = 1$$

$$\beta = B\tan\theta$$

$$\gamma = C\tan^2\theta$$

$$\delta = D\tan^3\theta$$

Then,

$$B = \frac{\beta}{\tan\theta}$$

$$C = \frac{\gamma}{\tan^2\theta}$$

$$D = \frac{\delta}{\tan^3\theta}$$

Furthermore:

$$\zeta(C+3D)\tan^2\theta-(B-3D)\tan\theta-\zeta(D+1)=0 \qquad \textit{[Ref. Equation 30]}$$

$$\tan^2\theta+\frac{3D-B}{\zeta(C+3D)}\tan\theta-\frac{\zeta(D+1)}{\zeta(C+3D)}=0$$

Substitution renders,

$$\tan^2\theta+\frac{\frac{3\delta}{\tan^3\theta}-\frac{\beta}{\tan\theta}}{\zeta(\frac{\gamma}{\tan^2\theta}+\frac{3\delta}{\tan^3\theta})}\tan\theta-\frac{\zeta(\frac{\delta}{\tan^3\theta}+1)}{\zeta(\frac{\gamma}{\tan^2\theta}+\frac{3\delta}{\tan^3\theta})}=0$$

$$\tan^2\theta+\frac{3\delta-\beta\tan^2\theta}{\zeta(\gamma\tan\theta+3\delta)}\tan\theta-(\frac{\delta+\tan^3\theta}{\gamma\tan\theta+3\delta})=0$$

<u>Verification</u>:

$$\tan^3\theta=3\tan\theta-\tan(3\theta)(1-3\tan^2\theta) \qquad \textit{[Ref. Equation 3]}$$

$$\tan^3\theta=3\tan\theta-\zeta(1-3\tan^2\theta)$$

$$\tan^3\theta-3\zeta\tan^2\theta-3\tan\theta+\zeta=0 \qquad \textit{[3θ Cubic Equation]}$$

Where,

$$\alpha z^3+\beta z^2+\gamma z+\delta=0 \qquad \textit{[Ref. Equation 32]}$$

Matching like terms renders:

$$\alpha=1$$

$$\beta=-3\zeta$$

$$\gamma=-3$$

$$\delta=\zeta$$

Substitution into the ***general expression*** derived above gives,

$$\tan^2\theta+\frac{3\delta-\beta\tan^2\theta}{\zeta(\gamma\tan\theta+3\delta)}\tan\theta-(\frac{\delta+\tan^3\theta}{\gamma\tan\theta+3\delta})=0$$

$$\tan^2\theta+\frac{3\zeta+3\zeta\tan^2\theta}{\zeta(-3\tan\theta+3\zeta)}\tan\theta-(\frac{\zeta+\tan^3\theta}{-3\tan\theta+3\zeta})=0$$

Now,

$$(x-x_1)(x-x_2)=0$$

$$x^2-(x_1+x_2)x+x_1x_2=0$$

Comparison of third terms yields:

$$x_1 x_2 = -(\frac{\zeta + \tan^3\theta}{-3\tan\theta + 3\zeta})$$

$$\tan\theta(x_2) = -(\frac{\zeta + \tan^3\theta}{-3\tan\theta + 3\zeta})$$

$$x_2 = -[\frac{\zeta + \tan^3\theta}{\tan\theta(-3\tan\theta + 3\zeta)}]$$

$$= -[\frac{3\tan\theta + 3\zeta\tan^2\theta}{\tan\theta(-3\tan\theta + 3\zeta)}]$$

$$= -[\frac{3\tan\theta(1 + \zeta\tan\theta)}{3\tan\theta(\zeta - \tan\theta)}]$$

$$= -(\frac{1 + \zeta\tan\theta}{\zeta - \tan\theta})$$

$$= -[\frac{1}{\tan(3\theta - \theta)}]$$

$$= -\frac{1}{\tan(2\theta)} \qquad \text{Q.E.D.}$$

PROBLEM NUMBER 19 *(Ref. Section 11.2)*

GIVEN:

The following *equation*:

$$z^3 - 1.160918746z^2 + 0.340299284z - 0.018283588 = 0$$

DEMONSTRATE:

An accelerated resolution approach:

SOLUTION:

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \qquad \text{[Ref. Equation 36]}$$

$$= \frac{-0.018283588 + 1.160918746}{1 - 0.340299284}$$

$$= \frac{1.142635157}{0.659700716}$$

$$= \sqrt{3}$$

$$3\theta = 60^o$$

$$\theta = 20^o$$

$$\tan\theta = 0.363970234$$

$$\tan^2\theta = 0.132474331$$

$$\tan^3\theta = 0.048216713$$

For the *Generalized Cubic Equation* written below:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \text{[Ref. Equation 32]}$$

$$z^3 - 1.160918746z^2 + 0.340299284z - 0.018283588 = 0$$

Where,

$$B\tan\theta = \beta$$

$$C\tan^2\theta = \gamma$$

$$D\tan^3\theta = \delta$$

Then,

$$B = \frac{\beta}{\tan\theta}$$

$$= \frac{-1.160918746}{0.363970234}$$

$$= -3.189598041$$

$$C = \frac{\gamma}{\tan^2 \theta}$$
$$= \frac{0.340299284}{0.132474331}$$
$$= 2.568794123$$

$$D = \frac{\delta}{\tan^3 \theta}$$
$$= \frac{-0.018283588}{0.048216713}$$
$$= -0.379196082$$

Hence, the *associated Characteristic Cubic Equation* is as follows, where "A" equals unity:

$$AR^3 + BR^2 + CR + D = 0$$

$$R^3 - 3.189598041R^2 + 2.568794123R - 0.379196082 = 0$$

The summation of coefficients is as follows:

$$A + B + C + D = 1 - 3.189598041 + 2.568794123 - 0.379196082$$
$$= 0$$

For this to occur, R can equal unity since:

$$(1)^3 - 3.189598041(1)^2 + 2.568794123(1) - 0.379196082 = 0$$
$$1 - 3.189598041 + 2.568794123 - 0.379196082 = 0$$

Or,

$$A + B + C + D = 0$$

Therefore,

$$z_R = R \tan \theta$$
$$= 1 \tan \theta$$
$$= \tan \theta$$

Or,

$$z_R{}^3 - 1.160918746 z_R{}^2 + 0.340299284 z_R - 0.018283588 = 0$$
$$(0.363970234)^3 - 1.160918746(0.363970234)^2 + 0.340299284(0.363970234) - 0.018283588 = 0$$
$$0.048216713 - 1.160918746(0.132474331) + 0.340299284(0.363970234) - 0.018283588 = 0$$
$$0.048216713 - 0.153791934 + 0.12385881 - 0.018283588 = 0$$
$$0 = 0$$

The two remaining roots are determined as follows:

Where,

$$
\begin{aligned}
B &= -(R+S+T) \\
&= -(1+S+T) \\
&= -3.189598041
\end{aligned}
$$

Or,

$$
\begin{aligned}
-3.189598041 &= -(1+S+T) \\
T &= 3.189598041-(1+S)
\end{aligned}
$$

$$
\begin{aligned}
C &= RS+RT+ST \\
&= S+T+ST \\
&= 2.568794123
\end{aligned}
$$

Or,

$$
\begin{aligned}
2.568794123 &= S+T+ST \\
&= S+3.189598041-(1+S)+S(3.189598041-1-S) \\
&= +2.1895980411+S(2.189598041-S)
\end{aligned}
$$

$$S^2-2.189598041S+0.379196082=0$$

Which assumes the form,

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

Where,

$$
\begin{aligned}
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\
&= \frac{(-b/a)\pm(1/a)\sqrt{b^2-4ac}}{2} \\
&= \frac{(-b/a)\pm\sqrt{(b/a)^2-4(c/a)}}{2} \\
&= \frac{(2.189598041)\pm\sqrt{(-2.189598041)^2-4(0.379196082)}}{2} \\
&= \frac{(2.189598041)\pm\sqrt{4.794339581-1.516784328}}{2} \\
&= \frac{(2.189598041)\pm\sqrt{3.277555253}}{2} \\
&= \frac{(2.189598041)\pm1.810401959}{2}
\end{aligned}
$$

$$S;T=2;0.189598041$$

Then,

$$\begin{aligned} z_S &= S\tan\theta \\ &= 2\tan\theta \\ &= 0.727940468 \end{aligned}$$

Or,

$$z_S{}^3 - 1.160918746 z_S{}^2 + 0.340299284 z_S - 0.018283588 = 0$$
$$(0.727940468)^3 - 1.160918746(0.727940468)^2 + 0.340299284(0.727940468) - 0.018283588 = 0$$
$$0.385733707 - 0.615167738 + 0.24771762 - 0.018283588 = 0$$
$$0.018283588 - 0.018283588 = 0$$
$$0 = 0$$

$$\begin{aligned} z_T &= T\tan\theta \\ &= 0.189598041\tan\theta \\ &= 0.069008043 \end{aligned}$$

Or,

$$z_S{}^3 - 1.160918746 z_S{}^2 + 0.340299284 z_S - 0.018283588 = 0$$
$$(0.069008043)^3 - 1.160918746(0.069008043)^2 + 0.340299284(0.069008043) - 0.018283588 = 0$$
$$0.000328623 - 0.005528422 + 0.023483387 - 0.018283588 = 0$$
$$0.018283588 - 0.018283588 = 0$$
$$0 = 0$$

In conclusion, this analysis demonstrates that any *Generalized Cubic Equation* whose *associated Characteristic Cubic Equation coefficients* sum to zero, can be easily resolved!

PROBLEM NUMBER 20 *(Ref. Section 11.3)*

GIVEN:

The following *Generalized Cubic Equation*:

$$z^3 - 1.309401077z^2 + 0.154700538z + 0.154700538 = 0$$

RESOLVE:

The above given equation

SOLUTION:

For the following Generalized Cubic Equation (*Ref. Equation 32*:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \text{[Ref. Equation 32]}$$

$$z^3 - 1.309401077z^2 + 0.154700538z + 0.154700538 = 0$$

Where,

$$\alpha = 1$$
$$\beta = -1.309401077$$
$$\gamma = +0.154700538$$
$$\delta = +0.154700538$$
$$\Sigma = 0$$

Hence:

$$z_R = 1$$

Verification:

$$z^3 - 1.309401077z^2 + 0.154700538z + 0.154700538 = 0$$
$$1 - 1.309401077 + 0.154700538 + 0.154700538 = 0$$
$$-0.309401077 + 2(0.154700538) = 0$$
$$-0.309401077 + 0.309401077 = 0$$
$$0 = 0$$

Or,

$$\alpha + \beta + \gamma + \delta = 0$$

Then,

$$(z - z_R)(z - z_S)(z - z_T) = 0$$
$$(z - z_R)[z^2 - (z_S + z_T)z + z_S z_T] = 0$$
$$z^3 - (z_R + z_S + z_T)z^2 + [z_R(z_S + z_T) + z_S z_T]z - z_R z_S z_T = 0$$
$$z^3 - (1 + z_S + z_T)z^2 + [(1)(z_S + z_T) + z_S z_T]z - (1)z_S z_T = 0$$
$$z^3 - (1 + z_S + z_T)z^2 + (z_S + z_T + z_S z_T)z - z_S z_T = 0$$

But,

$$z^3 - 1.309401077z^2 + 0.154700538z + 0.154700538 = 0$$

So,

$$1 + z_S + z_T = 1.309401077$$
$$z_S + z_T + z_S z_T = 0.154700538$$
$$-z_S z_T = 0.154700538$$

Therefore,

$$z_S = 1.309401077 - (1 + z_T)$$
$$= 0.309401077 - z_T$$

And,

$$-z_S z_T = 0.154700538$$
$$-(0.309401077 - z_T)z_T = 0.154700538$$
$$z_T{}^2 - 0.309401077 z_T - 0.154700538 = 0$$

Where

$$az^2 + bz + c = 0$$

Such that,

$$a = 1$$
$$b = -0.309401077$$
$$c = -0.154700538$$

Then, via *Quadratic Formula*:

$$z_S; z_T = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$$
$$= \frac{1}{2(1)}[+0.309401077 \pm \sqrt{(-0.309401077)^2 - 4(1)(-0.154700538)}] \backslash$$
$$= \frac{1}{2}(+0.309401077 \pm \sqrt{0.714531178})$$
$$= \frac{1}{2}(+0.309401077 \pm 0.84529946)$$
$$= 0.577350269; -0.267949192$$

First Check:

$$z^3 - 1.309401077z^2 + 0.154700538z + 0.154700538 = 0$$
$$(0.577350269)^3 - 1.309401077(0.577350269)^2 + 0.154700538(0.577350269) + 0.154700538 = 0$$
$$0.192450089 - 0.436467025 + 0.089316397 + 0.154700538 = 0$$
$$0 = 0$$

$$z^3 - 1.309401077z^2 + 0.154700538z + 0.154700538 = 0$$
$$(-0.267949192)^3 - 1.309401077(0.577350269)^2 + 0.154700538(-0.267949192) + 0.154700538 = 0$$
$$0.019237886 - 0.094010767 - 0.041451884 + 0.154700538 = 0$$
$$0 = 0$$

Second Check:

For the above given equation:

$$z^3 - 1.309401077z^2 + 0.154700538z + 0.154700538 = 0$$

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \qquad \textit{[Ref. Equation 36]}$$

$$= \frac{+0.154700538 + 1.309401077}{1 - 0.154700538}$$

$$= 1.732050808$$

$$= \sqrt{3}$$

$$= \tan(3\theta)$$

$$\tan(3\theta) = \tan 60^o$$

Or,

$$3\theta = 60^o$$

Then,

$$z_R = 1 = \tan\theta_R = \tan 45^o$$

$$\theta_R = 45^o$$

$$z_S = 0.577350269 = \frac{1}{\zeta} = \tan\theta_S = \tan 30^o$$

$$\theta_S = 30^o$$

$$z_T = -0.267949192 = \zeta - 2 = \tan\theta_T = \tan(-15^o)$$

$$\theta_T = -15^o$$

So,

$$\theta_R + \theta_S + \theta_T = 45^o + 30^o - 15^o$$

$$= 60^o$$

$$= 3\theta \qquad \textit{Q.E.D.}$$

PROBLEM NUMBER 21 *(Ref. Section 11.3)*

GIVEN:

The *Known Cubic Equation for the Tangent* (3θ) when $\zeta = \sqrt{3}$ *(Ref. Equation 1)*

DETERMINE:

A *Cubic Equation* which can be used in conjunction with the above *given equation* in order to simultaneously resolve it

SOLUTION:

Since tan θ represents the first root of the *Known Cubic Equation for the Tangent* (3θ) *(Ref. Equation 3)*:

For

$$\zeta = \sqrt{3} = \tan(3\theta) = \tan 60^o$$

$$3\theta = 60^o$$
$$\theta = 20^o$$

$$\tan\theta = \tan 20^o = 0.363970234$$

$$z_R = R\tan\theta = 1\tan\theta = \tan\theta = \tan\theta_R = \tan 20^o = 0.36370234$$

$$\theta = \theta_R = 20^o$$

Where *(Ref. Section 10)*,

$$3\theta = \theta_R + \theta_S + \theta_T$$
$$3\theta = \theta + \theta_S + \theta_T$$
$$2\theta = \theta_S + \theta_T$$
$$2\theta - \theta_S = \theta_T$$

When θ_S is represented by one randomly selected value, then θ_T may be calculated using the equation above. So when:

$$\theta_S = 30.38992734^o$$
$$\tan\theta_S = \tan 30.38992734^o$$
$$= 0.586460225$$
$$z_S = 0.586460225$$

$$\theta_T = 2\theta - \theta_S$$
$$= 2\theta - 30.38992734^o$$
$$= 2(20^o) - 30.38992734^o$$
$$= 9.610072646^o$$

$$\tan\theta_T = \tan 9.610072646^o$$
$$= 0.169318175$$

$$z_T = 0.169318175$$

The resulting *Generalized Cubic Equation (Ref. Equation 32)* is determined as follows:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \textit{[Ref. Equation 32]}$$

Where,

$$(z - z_R)(z - z_S)(z - z_T) = 0$$

$$z^3 - (z_R + z_S + z_T)z^2 + (z_R z_S + z_R z_T + z_S z_T)z - z_R z_S z_T = 0$$

Comparing like coefficients renders,

$$\beta = -(z_R + z_S + z_T) = -(0.363970234 + 0.586460225 + 0.169318175)$$
$$= -1.119748634$$

$$\gamma = z_R(z_S + z_T) + z_S z_T = 0.363970234(0.586460225 + 0.169318175) + (0.586460225)(0.169318175)$$
$$= 0.374379216$$

$$\delta = -z_R z_S z_T = -0.363970234(0.586460225)(0.169318175)$$
$$= -0.036141652$$

Then,

$$z^3 - 1.119748634z^2 + 0.374379216z - 0.036141652 = 0$$

Lastly, the *cubic term* of this equation may be substituted for by the *cubic term relationship* expressed in the *Known Cubic Equation for the Tangent (3θ) (Ref. Equation 3)* below:

Where,

$$\tan^3\theta = 3\tan\theta - \zeta(1 - 3\tan^2\theta) \quad \textit{[Ref. Equation 3]}$$

Or,

$$z^3 = 3z - \zeta(1 - 3z^2)$$

For $\zeta = \sqrt{3}$,

$$z^3 = 3z \quad \sqrt{3}(1 \quad 3z^2)$$

Substitution occurs as follows:

$$z^3 - 1.119748634z^2 + 0.374379216z - 0.036141652 = 0$$
$$[3z - \sqrt{3}(1 - 3z^2)] - 1.119748634z^2 + 0.374379216z - 0.036141652 = 0$$
$$3\sqrt{3}z^2 + 3z - \sqrt{3} - 1.119748634z^2 + 0.374379216z - 0.036141652 = 0$$
$$4.076403789z^2 + 3.374379216z - 1.76819246 = 0$$

$$z_1, z_2 = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$$

$$= \frac{1}{2(4.076403789)}[-(3.374379216) \pm \sqrt{(3.374379216)^2 + 4(4.076403789)(1.76819246)}]$$

$$= \frac{1}{2(4.076403789)}[-(3.374379216) \pm 6.3417585]$$

$$= 0.363970234; -1.191753592$$

$$= \tan\theta; -\frac{1}{\tan(2\theta)}$$

Hence, the two equations are *simultaneously resolved* by virtue of combining the two following independent equations:

$$z^3 - 1.119748634z^2 + 0.374379216z - 0.036141652 = 0$$

$$z^3 = 3z - \sqrt{3}(1 - 3z^2)$$

The *root* which they contain in common is as follows, where the other root is considered to be extraneous.

$$z_R = 0.363970234$$

The other two roots for the *first equation above* are determined by using the following relationship:

$$(z - z_R)(z - z_S)(z - z_T) = 0$$

$$(z - z_R)[z^2 - (z_S + z_T)z + z_S z_T] = 0$$

$$(z - z_R)(z^2 + Mz + N) = 0$$

$$z^3 + (M - z_R)z^2 + (N - Mz_R)z - Nz_R = 0$$

Where,

$$z^3 - 1.119748634z^2 + 0.374379216z - 0.036141652 = 0$$

Comparing like coefficients gives:

$$-1.119748634 = M - z_R$$

$$0.374379216 = N - Mz_R$$

$$-0.036141652 = -Nz_R$$

Then,

$$M = z_R - 1.119748634 = 0.363970234 - 1.119748634 = -0.755778399$$

$$N = \frac{0.036141652}{z_R} = \frac{0.036141652}{0.363970234} = 0.099298372$$

Check,

$$0.374379216 = N - Mz_R = 0.099298372 - (-0.755778399)(0.363970234)$$

For the factor $(z^2 + Mz + N) = 0$, resolution occurs by applying the *Quadratic Formula* as follows:

$$z_S, z_T = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$$
$$= \frac{1}{2(1)}[-M \pm \sqrt{(M)^2 - 4(1)N}]$$
$$= \frac{1}{2(1)}[-(-0.755778399) \pm \sqrt{(-0.755778399)^2 - 4(1)(0.099298372)}]$$
$$= \frac{1}{2(1)}(0.755778399) \pm \sqrt{0.571200988 - 0.397193488})$$
$$= \frac{1}{2(1)}(0.755778399) \pm \sqrt{0.1740075})$$
$$= \frac{1}{2}(0.755778399) \pm 0.417142062)$$
$$= \frac{1}{2}(0.755778399) \pm 0.417142062)$$
$$= 0.58646023; 0.169318168$$

Check,

$$z_R^3 - 1.119748634 z_R^2 + 0.374379216 z_R - 0.036141652 = 0$$
$$(0.363970234)^3 - 1.119748634(0.363970234)^2 + 0.374379216(0.363970234) - 0.036141652 = 0$$
$$0.048216713 - 0.148337951 + 0.136262891 - 0.036141652 = 0$$
$$0.036141652 - 0.036141652 = 0$$
$$0 = 0$$

$$z_S^3 - 1.119748634 z_S^2 + 0.374379216 z_S - 0.036141652 = 0$$
$$(0.58646023)^3 - 1.119748634(0.58646023)^2 + 0.374379216(0.58646023) - 0.036141652 = 0$$
$$0.201704551 - 0.385121419 + 0.219558521 - 0.036141652 = 0$$
$$0.036141652 - 0.036141652 = 0$$
$$0 = 0$$

$$z_T^3 - 1.119748634 z_T^2 + 0.374379216 z_T - 0.036141652 = 0$$
$$(0.169318168)^3 - 1.119748634(0.169318168)^2 + 0.374379216(0.169318168) - 0.036141652 = 0$$
$$0.004854121 - 0.032101672 + 0.063389202 - 0.036141652 = 0$$
$$0.036141652 - 0.036141652 = 0$$
$$0 = 0$$

The remaining two roots for the *second equation above*, otherwise referred to as the *Known Cubic Equation for the Tangent (3θ) (Ref. Equation 3)* already have been determined as follows *(Ref. Section 2.4.3)*:

$$z_S = \tan(\theta + 120^o) = \tan(20^o + 120^o) = \tan 140^o = \tan(180^o - 140^o) = -\tan(40^o) = -\tan(2\theta)$$
$$z_T = \tan(\theta + 240^o) = \tan(20^o + 240^o) = \tan 260^o = \tan(180^o + 80^o) = \tan(80^o) = \tan(4\theta)$$

PROBLEM NUMBER 22 *(Ref. Sections 11.3 and 11.6)*

GIVEN:

The following *Generalized Cubic Equation*:

$$z^3 - \frac{19}{18}z^2 + \frac{5}{18}z - \frac{1}{81} = 0$$

RESOLVE:

The above given equation

SOLUTION:

For,

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 \quad \frac{19}{18}z^2 + \frac{5}{18}z \quad \frac{1}{81} = 0$$

Where,

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \qquad \textit{[Ref. Equation 36]}$$

$$\tan(3\theta) = \frac{-\frac{1}{81} + \frac{19}{18}}{1 - \frac{5}{18}}$$

$$= \frac{-(\frac{2}{2})[\frac{1}{9(9)}] + (\frac{9}{9})[\frac{19}{9(2)}]}{(\frac{162}{162})(1) - (\frac{9}{9})[\frac{5}{9(2)}]}$$

$$= \frac{-\frac{2}{162} + \frac{171}{162}}{\frac{162}{162} - \frac{45}{162}}$$

$$= \frac{-2 + 171}{162 - 45}$$

$$= \frac{169}{117}$$

$$= \frac{13(13)}{9(13)}$$

$$= \frac{13}{9}$$

$$3\theta = 55.30484647^o$$
$$\theta = 18.43494882^o$$

$$\tan\theta = \frac{1}{3}$$

The associated *Characteristic Cubic Equation* is determined as follows:

Where,

$$z_R = R\tan\theta$$

And

$$\beta = B\tan\theta$$
$$\gamma = C\tan^2\theta$$
$$\delta = D\tan^3\theta$$

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \textit{[Ref. Equation 32]}$$

$$(R\tan\theta)^3 + (B\tan\theta)(R\tan\theta)^2 + (C\tan^2\theta)(R\tan\theta) + (D\tan^3\theta) = 0$$

$$R^3 + BR^2 + CR + D = 0$$

Where,

$$\beta = -\frac{19}{18} = B\tan\theta = \frac{B}{3}$$
$$\gamma = \frac{5}{18} = C\tan^2\theta = \frac{C}{9}$$
$$\delta = -\frac{1}{81} = D\tan^3\theta = \frac{D}{27}$$

Or,

$$B = 3\beta = -3(\frac{19}{18}) = -\frac{19}{6}$$
$$C = 9\gamma = 9(\frac{5}{18}) = \frac{5}{2}$$
$$D = 27\delta = -27(\frac{1}{81}) = -\frac{1}{3}$$

Then

$$R^3 + BR^2 + CR + D = 0,$$
$$R^3 - \frac{19}{6}R^2 + \frac{5}{2}R - \frac{1}{3} = 0$$

Where the sum of the coefficients is as follows:

$$A + B + C + D = 1 - \frac{19}{6} + \frac{5}{2} - \frac{1}{3}$$
$$= \frac{1}{6}(6 - 19 + 15 - 2)$$
$$= 0$$

Hence,

$$R = 1$$

Verification:

$$R^3 - \frac{19}{6}R^2 + \frac{5}{2}R - \frac{1}{3} = 0$$

$$(1)^3 - \frac{19}{6}(1)^2 + \frac{5}{2}(1) - \frac{1}{3} = 0$$

$$1 - \frac{19}{6} + \frac{5}{2} - \frac{1}{3} = 0 \qquad \textit{(Same result as above)}$$

Then

$$z_R = R\tan\theta = (1)\tan\theta = \frac{1}{3}$$

So,

$$(z - z_R)(z - z_S)(z - z_T) = 0$$

$$(z - z_R)[z^2 - (z_S + z_T)z + z_S z_T] = 0$$

$$z^3 - (z_R + z_S + z_T)z^2 + [z_R(z_S + z_T) + z_S z_T]z - z_R z_S z_T = 0$$

$$z^3 - (\frac{1}{3} + z_S + z_T)z^2 + [\frac{1}{3}(z_S + z_T) + z_S z_T]z - \frac{1}{3}z_S z_T = 0$$

But,

$$z^3 - \frac{19}{18}z^2 + \frac{5}{18}z - \frac{1}{81} = 0$$

Comparing like coefficients gives,

$$\frac{1}{3} + z_S + z_T = \frac{19}{18}$$

$$\frac{1}{3}(z_S + z_T) + z_S z_T = \frac{5}{18}$$

$$\frac{1}{3}z_S z_T = \frac{1}{81}$$

Therefore,

$$z_S = \frac{19}{18} - (\frac{6}{18} + z_T)$$

$$= \frac{13}{18} - z_T$$

And,

$$z_S z_T = \frac{3}{81}$$

$$(\frac{13}{18} - z_T)z_T = \frac{1}{27}$$

$$z_T{}^2 - \frac{13}{18} z_T + \frac{1}{27} = 0$$

Where

$$az^2 + bz + c = 0$$

Such that,

$$a = 1$$

$$b = -\frac{13}{18}$$

$$c = \frac{1}{27}$$

Via *Quadratic Formula*:

$$z_S ; z_T = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$$

$$= \frac{1}{2(1)}[\frac{13}{18} \pm \sqrt{(-\frac{13}{18})^2 - 4(1)(\frac{1}{27})}]$$

$$= \frac{1}{2(1)}\{\frac{13}{18} \pm \sqrt{(-\frac{13}{18})^2 - 4(1)[\frac{1}{9(3)}][\frac{12}{4(3)}]}\}$$

$$= \frac{1}{2(1)}(\frac{13}{18} \pm \frac{1}{18}\sqrt{169 - 48})$$

$$= \frac{1}{36}(13 \pm \sqrt{121})$$

$$= \frac{1}{36}(13 \pm 11)$$

$$= \frac{1}{36}(24 \pm 2)$$

$$= \frac{2}{3} ; \frac{1}{18}$$

First Check,

$$z_R{}^3 - \frac{19}{18} z_R{}^2 + \frac{5}{18} z_R - \frac{1}{81} = 0$$

$$(\frac{1}{3})^3 - \frac{19}{18}(\frac{1}{3})^2 + \frac{5}{18}(\frac{1}{3}) - \frac{1}{81} = 0$$

$$\frac{1}{3(9)} - \frac{19}{18(9)} + \frac{5}{18(3)} - \frac{1}{9(9)} = 0$$

$$\frac{1}{3(9)}(\frac{6}{6}) - \frac{19}{18(9)} + \frac{5}{6(9)}(\frac{3}{3}) - \frac{1}{9(9)}(\frac{2}{2}) = 0$$

$$\frac{6 - 19 + 15 - 2}{162} = 0$$

$$\frac{21-21}{162}=0$$
$$0=0$$

$$z_S{}^3-\frac{19}{18}z_S{}^2+\frac{5}{18}z_S-\frac{1}{81}=0$$
$$(\frac{2}{3})^3-\frac{19}{18}(\frac{2}{3})^2+\frac{5}{18}(\frac{2}{3})-\frac{1}{81}=0$$
$$\frac{8}{3(9)}-\frac{76}{18(9)}+\frac{10}{18(3)}-\frac{1}{9(9)}=0$$
$$\frac{8}{3(9)}(\frac{6}{6})-\frac{76}{18(9)}+\frac{10}{6(9)}(\frac{3}{3})-\frac{1}{9(9)}(\frac{2}{2})=0$$
$$\frac{48-76+30-2}{162}=0$$
$$\frac{78-78}{162}=0$$
$$0=0$$

$$z_T{}^3-\frac{19}{18}z_T{}^2+\frac{5}{18}z_T-\frac{1}{81}=0$$
$$(\frac{1}{18})^3-\frac{19}{18}(\frac{1}{18})^2+\frac{5}{18}(\frac{1}{18})-\frac{1}{81}=0$$
$$(\frac{1-19}{18^3})+\frac{5}{18}(\frac{1}{18})-\frac{1}{81}=0$$
$$\frac{-18}{18^3}+\frac{5}{18}(\frac{1}{18})-\frac{1}{81}=0$$
$$-\frac{1}{18^2}+\frac{5}{18^2}-\frac{1}{81}=0$$
$$\frac{4}{18^2}-\frac{1}{81}=0$$
$$(\frac{2}{18})^2-\frac{1}{81}=0$$
$$(\frac{1}{9})^2-\frac{1}{81}=0$$
$$\frac{1}{81}-\frac{1}{81}=0$$
$$0=0$$

Second Check,

Since $R=1$, this equation may be compared to its associated *3θ Cubic Equation* as follows:

$$z^3 - 3\zeta z^2 - 3z + \zeta = 0$$

$$z^3 - 3(\frac{13}{9})z^2 - 3z + \frac{13}{9} = 0$$

$$z^3 - (\frac{39}{9})z^2 - 3z + \frac{13}{9} = 0$$

Where,

$$z^3 = \frac{19}{18}z^2 - \frac{5}{18}z + \frac{1}{81}$$

This *quadratic expression* for z^3 then is substituted into the associated *3θ Cubic Equation* result in order to produce:

$$(\frac{19}{18}z^2 - \frac{5}{18}z + \frac{1}{81}) - (\frac{39}{9})z^2 - 3z + \frac{13}{9} = 0$$

$$(\frac{19-78}{18})z^2 - (\frac{5+54}{18})z + \frac{1+117}{81} = 0$$

$$-\frac{59}{18}z^2 - \frac{59}{18}z + \frac{118}{81} = 0$$

$$-\frac{59(9)}{9(2)(9)}z^2 - \frac{59(9)}{9(2)(9)}z + \frac{118(2)}{9(9)(2)} = 0$$

$$-59(9)z^2 - 59(9)z + 118(2) = 0$$

$$z^2 + z - \frac{118(2)}{59(9)} = 0$$

$$z^2 + z - \frac{59(2)(2)}{59(9)} = 0$$

$$z^2 + z - \frac{4}{9} = 0$$

$$z^2 + z = \frac{4}{9}$$

$$z^2 + z + (\frac{1}{2})^2 = \frac{4}{9} + (\frac{1}{2})^2$$

$$(z + \frac{1}{2})^2 = \frac{4}{9} + \frac{1}{4}$$

$$z + \frac{1}{2} = \sqrt{\frac{4(4)}{9(4)} + \frac{1(9)}{4(9)}}$$

$$z = -\frac{1}{2} \pm \sqrt{\frac{25}{36}}$$
$$= \frac{-3 \pm 5}{6}$$
$$= \frac{1}{3}; -\frac{4}{3}$$

Hence, the following root again is verified for the above given equation:

$$z_R = \frac{1}{3}$$

PROBLEM NUMBER 23 *(Ref. Section 11)*

GIVEN:

The following RST Spread:

$$R = 2$$

$$S = 1$$

$$T = 0.189598041$$

DETERMINE:

a) The associated θ and 3θ

b) The associated following equations:

The Simplified Unified Cubic Trigonometric Reduction Equation	*Equation 30*
The Characteristic Cubic Equation	*Equation 31*
The Generalized Cubic Equation	*Equation 32*
The Cubic Restitution Equation	*Equation 35*
The ζ Relationship to Generalized Cubic Equation Coefficients	*Equation 36*

SOLUTION:

a) Where,

$$B = -(R + S + T)$$
$$= -(1 + 2 + T)$$
$$= -(3 + T)$$

$$C = RS + RT + ST$$
$$= 1(2) + (1)T + 2T$$
$$= 2 + 3T$$

$$D = -RST$$
$$= -1(2)T$$
$$= -2T$$

Using the *Determination of* $tan^2\theta$ *from Equation 31 Coefficients (Ref. Equation 37)*:

$$\tan^2\theta = \frac{3(B+C)+(D+1)\pm\sqrt{9(B^2+C^2)+D^2+14BC-6BD+6CD+1+6B-6C-34D}}{2C+6D} \quad \text{[Ref. Equation 30]}$$

$$= \frac{3(-3-T+2+3T)+(-2T+1)\pm\sqrt{9[(3+T)^2+(2+3T)^2]+4T^2-14(3+T)(2+3T)-6(3+T)(2T)-6(2+3T)(2T)+1-6(3+T)-6(2+3T)+68T}}{2(2+3T)-12T}$$

$$= \frac{3(2T-1)+(-2T+1)\pm\sqrt{9(13+18T+10T^2)+4T^2-14(6+11T+3T^2)-6(6T+2T^2)-6(4T+6T^2)+1-6(3+T)-6(2+3T)+68T}}{4+6T-12T}$$

$$= \frac{3(2T-1)+(-2T+1)\pm\sqrt{(117-84+1-18-12)+(162-154-36-24-6-18+68)T+(90+4-42-12-36)T^2}}{4-6T}$$

$$= \frac{4T-2\pm\sqrt{4-8T+4T^2}}{4-6T}$$

$$= \frac{4(0.189598041)-2\pm\sqrt{4-8(0.189598041)+4(0.189598041)^2}}{4-6(0.189598041)}$$

$$= \frac{4(0.189598041)-2\pm\sqrt{2.627005341}}{4-6(0.189598041)}$$

$$= \frac{-1.241607836\pm1.620803918}{4-6(0.189598041)}$$

$$= \frac{0.379196082}{2.862411754}, \frac{-2.862411754}{2.862411754}$$

$$= 0.13247433, -1$$

$$\tan\theta = \sqrt{0.13247433}; \sqrt{1}$$

$$= 0..363970234; imaginary$$

$$\theta = 20^o$$

$$3\theta = 60^o$$

b) Where,

[Ref. Equation 30] $\zeta[C+3D]\tan^2\theta-[B-3D]\tan\theta-\zeta(D+1)=0$

$$\tan 60^o[2+3T-6T]\tan^2\theta-[-3-T+6T]\tan\theta-\tan 60^o(-2T+1)=0$$

$$\sqrt{3}[2-3T]\tan^2\theta+[+3-5T]\tan\theta+\sqrt{3}(2T-1)=0$$

$$\sqrt{3}[2-3(0.189598041)]\tan^2\theta+[+3-5(0.189598041)]\tan\theta+\sqrt{3}(2(0.189598041)-1)=0$$

Hence, *Equation 30* may be written as,

$$2.478921295\tan^2\theta+2.052009795\tan\theta-1.075263928=0$$

Where,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{5T-3\pm\sqrt{(3-5T)^2+4\sqrt{3}^2(2-3T)(1-2T)}}{2\sqrt{3}(2-3T)}$$

$$=\frac{5T-3\pm\sqrt{9-30T+25T^2+12(2-7T+6T^2)}}{\sqrt{3}(4-6T)}$$

$$=\frac{5T-3\pm\sqrt{33-114T+97T^2}}{\sqrt{3}(4-6T)}$$

$$=\frac{5(0.189598041)-3\pm\sqrt{33-114(0.189598041)+97(0.189598041)^2}}{\sqrt{3}[4-6(0.189598041)]}$$

$$=\frac{-2.052009795\pm\sqrt{14.87272279}}{4.95784259}$$

$$=\frac{-2.052009795\pm 3.856516925}{4.95784259}$$

$$=\frac{+1.80450713}{4.95784259},\frac{-5.90852672}{4.95784259}$$

$$=0.363970234,-1.191753593$$

$$=\tan 20^o,-\tan 50^o$$

$$=\tan\theta,-\frac{1}{\tan(2\theta)}$$

For this condition, the value for T is verified as follows:

$$z_R=R\tan\theta=\tan\theta_R=(2)\tan 20^o$$

$$=(2)0.363970234$$

$$=0.727940468$$

$$=\tan 36.05238873^o$$

$$\theta_R=36.05238873^o$$

$$z_S = S\tan\theta = \tan\theta_S = (1)\tan 20^o$$

$$\theta_R = 20^o$$

Where,

$$\theta_R + \theta_S + \theta_T = 60^o$$

$$\theta_T = 60^o - (\theta_R + \theta_S)$$

$$= 60^o - (36.05238873^o + 20^o)$$

$$= 60^o - 56.05238873^o$$

$$= 3.94761127^o$$

And,

$$z_T = \tan\theta_T = T\tan\theta = \tan 3.94761127^o$$

$$T = \frac{\tan 3.94761127^o}{\tan 20^o}$$

$$T = \frac{0.069008043}{0.363970234}$$

$$= 0.189598041$$ Q.E.D.

For,

$$AR^3 + BR^2 + CR + D = 0$$ *[Ref. Equation 31]*

Where,

$$A = 1$$

$$B = -(3+T)$$
$$= -3.189598041$$

$$C = 2 + 3T$$
$$= 2.568794123$$

$$D = -2T$$
$$= -0.379196082$$

Then,

$$R^3 - 3.189598041R^2 + 2.568794123R - 0.379196082 = 0$$

The associated *Generalized Cubic Equation (Ref. Equation 32)* is determined as follows,

Where,

$$\alpha = 1$$
$$\beta = B\tan\theta$$
$$\gamma = C\tan^2\theta$$
$$\delta = D\tan^3\theta$$

[Ref. Equation 32] $$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$$

$$1z^3 + (B\tan\theta)z^2 + (C\tan^2\theta)z + (D\tan^3\theta) = 0$$

$$z^3 - 3.189598041(\tan\theta)z^2 + 2.568794123(\tan^2\theta)z - 0.379196082\tan^3\theta = 0$$

$$z^3 - 3.189598041(\tan 20^o)z^2 + 2.568794123(\tan 20^o)^2 z - 0.379196082(\tan 20^o)^3 = 0$$

Then, *Equation 32* may be written as,

$$z^3 - 1.160918746z^2 + 0.340299284z - 0.018283588 = 0$$

Where,

$$z_R = R\tan\theta = 2\tan 20^o = 0.727940468$$
$$z_S = S\tan\theta = (1)\tan 20^o = 0.363970234$$
$$z_T = T\tan\theta = (0.189598041)\tan 20^o = 0.069008043$$

Checking these roots gives,

$$z_R{}^3 - 1.160918746z_R{}^2 + 0.340299284z_R - 0.018283588 = 0$$
$$0.727940468^3 - 1.160918746(0.727940468)^2 + 0.340299284(0.727940468) - 0.018283588 = 0$$
$$(0.385733707 - 0.615167738 + 0.24771762) - 0.018283588 = 0$$
$$0.018283588 - 0.018283588 = 0$$

$$z_S{}^3 - 1.160918746z_S{}^2 + 0.340299284z_S - 0.018283588 = 0$$
$$0.363970234^3 - 1.160918746(0.363970234)^2 + 0.340299284(0.363970234) - 0.018283588 = 0$$
$$(0.048216713 - 0.153791934 + 0.12385881) - 0.018283588 = 0$$
$$0.018283588 - 0.018283588 = 0$$

$$z_T{}^3 - 1.160918746z_T{}^2 + 0.340299284z_T - 0.018283588 = 0$$
$$0.069008043^3 - 1.160918746(0.069008043)^2 + 0.340299284(0.069008043) - 0.018283588 = 0$$
$$(0.000328623 - 0.005528422 + 0.023483387) - 0.018283588 = 0$$
$$0.018283588 - 0.018283588 = 0$$

Now, as a *first check*, the *ζ Relationship to Generalized Cubic Equation Coefficients (Ref. Equation 36)* is applied to the *Generalized Cubic Equation* result as follows:

Where,

$$z^3 - 1.160918746z^2 + 0.340299284z - 0.018283588 = 0$$

$$\zeta = \frac{\delta - \beta}{1 - \gamma}$$ *[Ref. Equation 36]*

$$= \frac{-0.018283588 + 1.160918746}{1 - 0.340299284}$$

$$= \frac{1.142635158}{0.659700716}$$

$$= 1.732050808$$

As a *further check, the Expression for S and T (Ref. Equation 33)* is employed as follows:

$$S, T = ½[-(B + R) \pm \sqrt{(B+R)^2 + \frac{4D}{R}}]$$ *[Ref. Equation 33]*

$$S, T = ½[-(-3 - T + 2) \pm \sqrt{(-1)^2 (T+1)^2 - \frac{8T}{2}}]$$

$$= ½[-(-T - 1) \pm \sqrt{(T+1)^2 - \frac{8T}{2}}]$$

$$= ½[T + 1) \pm \sqrt{(T+1)^2 - \frac{8T}{2}}]$$

$$= ½[1.189598041) \pm \sqrt{1.415143499 - .758392164}]$$

$$= ½[1.189598041) \pm \sqrt{0.656751335}]$$

$$= ½[1.189598041) \pm 0.810401959]$$

$$= ½[2, +0.379196082]$$

$$= 1, +0.189598041$$ Q.E.D.

For,

[Ref. Equation 35] $$D \tan^3\theta + \zeta C \tan^2\theta - B \tan\theta - \zeta = 0$$

$$-2T \tan^3\theta + \tan 60^o (2 + 3T) \tan^2\theta + (3 + T) \tan\theta - \tan 60^o = 0$$

$$-2T \tan^3\theta + \sqrt{3}(2 + 3T) \tan^2\theta + (3 + T) \tan\theta - \sqrt{3} = 0$$

Or, *Equation 35* may be written as,

$$-0.379196082 \tan^3\theta + 4.449281935 \tan^2\theta + 3.189598041 \tan\theta - 1.732050808 = 0$$

Check

$$-0.379196082 \tan(20^o)^3 + 4.449281935 \tan(20^o)^2 + 3.189598041 \tan(20^o) - 1.732050808 = 0$$

$$(-0.018283588 + 0.589415649 + 1.160918746) - 1.732050808 = 0$$

$$(1.732050808) - 1.732050808 = 0$$

<u>In summary</u>: For $R = 2$, $S = 1$, and $T = 0.189598041$, the *associated equation family* is as follows:

[Ref. Equation 30]
$$\zeta[C + 3D]\tan^2\theta - [B - 3D]\tan\theta - \zeta(D+1) = 0$$
$$2.478921295\tan^2\theta + 2.052009795\tan\theta - 1.075263928 = 0$$

[Ref. Equation 31]
$$AR^3 + BR^2 + CR + D = 0$$
$$R^3 - 3.189598041R^2 + 2.568794123R - 0.379196082 = 0$$

[Ref. Equation 32]
$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$$
$$z^3 - 1.160918746z^2 + 0.340299284z - 0.018283588 = 0$$

[Ref. Equation 35]
$$D\tan^3\theta + \zeta C\tan^2\theta - B\tan\theta - \zeta = 0$$
$$-0.379196082\tan^3\theta + 4.449281935\tan^2\theta + 3.189598041\tan\theta - 1.732050808 = 0$$

PROBLEM NUMBER 24 *(Ref. Section 11)*

GIVEN:

The *Characteristic Cubic Equation* below:

$$R^3 - 169.5305743R^2 + 133.6381482R - 51.11019143 = 0$$

CALCULATE:

tan θ.

SOLUTION:

Where,

$$AR^3 + BR^2 + CR + D = 0 \qquad \textit{[Ref. Equation 31]}$$

$$B = -169.5305743$$
$$C = 133.6381482$$
$$D = -51.11019143$$

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \qquad \textit{[Ref. Equation 36]}$$

And,

$$\tan^3\theta = 3\tan\theta - \zeta(1 - 3\tan^2\theta) \qquad \textit{[Ref. Equation 3]}$$

$$\tan^3\theta = 3\tan\theta + \zeta(3\tan^2\theta - 1)$$

$$\frac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1} = \zeta$$

So,

$$\zeta = \frac{\delta - \beta}{1 - \gamma} = \frac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$$

$$\frac{(D\tan^2\theta - B)\tan\theta}{1 - C\tan^2\theta} = \frac{(\tan^2\theta - 3)\tan\theta}{3\tan^2\theta - 1}$$

$$\frac{(D\tan^2\theta - B)}{1 - C\tan^2\theta} = \frac{\tan^2\theta - 3}{3\tan^2\theta - 1}$$

Multiplying thru by the product of the denominators gives:

$$(D\tan^2\theta - B)(3\tan^2\theta - 1) = (1 - C\tan^2\theta)(\tan^2\theta - 3)$$

$$3D\tan^4\theta - (3B + D)\tan^2\theta + B = -3 + (3C + 1)\tan^2\theta - C\tan^4\theta$$

$$(3D + C)\tan^4\theta - [3(B + C) + (D + 1)]\tan^2\theta + B + 3 = 0$$

Where,

$$\begin{aligned} a &= 3D + C \\ &= 3(-51.11019143) + 133.6381482 \\ &= -153.3305743 + 133.6381482 \\ &= -19.69242609 \end{aligned}$$

$$b = -[3(B + C) + (D + 1)]$$

$$= -[3(-169.5305743 + 133.6381482) + (-51.11019143 + 1)]$$
$$= -[3(-35.8924261) - 50.11019143]$$
$$= -(-107.6772783 - 50.11019143)$$
$$= +157.7874697$$
$$c = B + 3$$
$$= -169.5305743 + 3$$
$$= -166.5305743$$

The *Quadratic Formula* yields:

$$\tan^2\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-157.7874697 \pm \sqrt{(157.7874697)^2 - 4(-19.69242609)(-166.5305743)}}{2(-19.69242609)}$$
$$= \frac{-157.7874697 \pm \sqrt{24{,}896.6856 - 13{,}117.5641}}{2(-19.69242609)}$$
$$= \frac{-157.7874697 \pm \sqrt{11{,}779.3215}}{2(-19.69242609)}$$
$$= \frac{-157.7874697 \pm 108.5325827}{-39.38485218}$$
$$= 1.250604847$$

$$\tan\theta = \sqrt{1.250604847}$$
$$= 1.118304452$$

Check,

$$\theta = 48.19657146^o$$
$$3\theta = 144.5897144^o$$

$$\zeta = \tan(3\theta) = 0.710933225$$

$$\zeta = \frac{\delta - \beta}{1 - \gamma} = \frac{(D\tan^2\theta - B)\tan\theta}{1 - C\tan^2\theta}$$
$$= \frac{[(-51.11019143)(1.118304452)^2 + 169.5305743)](1.118304452)}{1 - (133.6381482)(1.118304452)^2}$$
$$= \frac{[(-63.91865315) + 169.5305743)](1.118304452)}{1 - 167.1285159}$$
$$= \frac{(105.6119212)(1.118304452)}{-166.1285159}$$
$$= \frac{118.1062816}{-166.1285159}$$
$$= -0.710933225$$

Q.E.D.

PROBLEM NUMBER 25 *(Ref. Section 13.2)*

GIVEN:

The following equation:

$$z_R = \frac{-\beta + \sqrt[3]{\beta^3 - 27\delta}}{3} \qquad \text{[Ref. Section 13.2]}$$

SHOW:

How the above *given equation* may be applied to resolve *the 3θ Cubic Equation.*

SOLUTION:

The *3θ Cubic Equation* is developed below:

$$\tan^3\theta = 3\tan\theta - \zeta(1 - 3\tan^2\theta) \qquad \text{[Ref. Equation 3]}$$

$$z^3 = 3z - \zeta(1 - 3z^2)$$

$$z^3 - 3\zeta z^2 - 3z + \zeta = 0$$

Where,

$$\zeta = \tan(3\theta)$$

$$\beta = -3\zeta$$

$$\beta^2 = 9\zeta^2$$

But, the above *given equation* applies when $\beta^2 = 3\alpha\gamma$. For the condition when $\alpha = 1$:

$$\beta^2 = 3\alpha\gamma$$
$$= 3(1)\gamma$$
$$= 3\gamma$$

Equating terms renders:

$$3\gamma = 9\zeta^2$$
$$\gamma = 3\zeta^2$$
$$\gamma z = 3\zeta^2 z$$

Adding and subtracting this amount to the *3θ Cubic Equation* doesn't change its outcome or summation as follows:

$$z^3 - 3\zeta z^2 + (3\zeta^2 z - 3\zeta^2 z) - 3z + \zeta = 0$$
$$z^3 - 3\zeta z^2 + 3\zeta^2 z + [\zeta - 3z(\zeta^2 + 1)] = 0$$

Or,

$$z^3 + \beta z^2 + \gamma z + \delta = 0$$

Where,

$$\delta = \zeta - 3z(\zeta^2 + 1)$$

And,

$$\beta = -3\zeta$$
$$\beta^2 = 9\zeta^2$$
$$= 3(3\zeta^2)$$
$$= 3\gamma$$

Hence, the *3θ Cubic Equation*, as modified above, *qualifies* for resolution via the above *given equation*. Therefore,

$$z_R = \frac{-\beta + \sqrt[3]{\beta^3 - 27\delta}}{3}$$
$$= \frac{3\zeta + \sqrt[3]{(-3\zeta)^3 - 27[\zeta - 3z(\zeta^2 + 1)]}}{3}$$

As a quick check, this result is tested for the following condition when:

$$3\theta = 60^o$$
$$\theta = 20^o$$
$$\zeta = \tan(3\theta) = \tan 60^o = \sqrt{3}$$
$$\tan\theta = \tan 20^o = 0.363970234$$
$$z_R = \frac{3\zeta + \sqrt[3]{(-3\zeta)^3 - 27[\zeta - 3z(\zeta^2 + 1)]}}{3}$$
$$= \frac{3\sqrt{3} + \sqrt[3]{(-3\sqrt{3})^3 - 27[\sqrt{3} - 3z_R(3+1)]}}{3}$$
$$= \frac{3\sqrt{3} + \sqrt[3]{(-3\sqrt{3})^3 - 27(\sqrt{3} - 12z_R)}}{3}$$
$$= \frac{3\sqrt{3} + \sqrt[3]{-27(\sqrt{3})^3 - 27[\sqrt{3} - 12(0.3639701234)]}}{3}$$
$$= \frac{3\sqrt{3} - 3\sqrt[3]{4\sqrt{3} - 12(0.3639701234)}}{3}$$
$$= \sqrt{3} - \sqrt[3]{6.92820323 - 4.367642811}$$
$$= \sqrt{3} - \sqrt[3]{2.560560419}$$
$$= 1.732050808 - 1.368080573$$
$$= 0.363970234 \qquad \textit{Q.E.D.}$$

Although this problem makes use of a *root* to determine *itself* (see above), it *more importantly* emphasizes the fact that equations may be *manipulated*, or ***transformed*** into other formats which avail certain meaningful attributes.

PROBLEM NUMBER 26 *(Ref. Sections 13.3.3 and 13.3.4)*

GIVEN:

$R^3 - 3.189598041R^2 + 2.568794123R - 0.379196082 = 0$

RESOLVE:

The above given equation

SOLUTION:

Substituting $R = u$ into the *above given equation* renders:

$$u^3 - 3.189598041u^2 + 2.568794123u - 0.379196082 = 0$$

Equation 42 is selected from the *Cubic Resolution Transform* excerpt since it matches the above format,

$$u^3 + (3V)u^2 + 3(V^2 - f^2)u + V^3 - 3f^2V - 2\psi f^3 = 0 \quad \textit{[Ref. Equation 42]}$$

Comparing respective coefficients returns the following results,

$$V = \frac{-3.189598041}{3} = -1.063199347$$

$$f = \sqrt{V^2 - \frac{2.568794123}{3}} = 0.523572481$$

$$\psi = \frac{V^3 - 3f^2V + 0.379196082}{2f^3} = 0.180182494 = \cos(6\omega)$$

Since $-1 \le \cos(6\omega) \le +1$ the *three roots are all real*. Hence,

$$6\omega = +79.61961026^o; -79.61961026^o; +(79.61961026^o + 360^o)$$
$$= +79.61961026^o; +280.3803897^o; +439.6196103^o$$

$$2\omega = \frac{+79.61961026^o}{3}; \frac{+280.3803897^o}{3}; \frac{+439.6196103^o}{3}$$
$$= +26.53987009^o; +93.46012991^o; +146.5398701^o$$

$$\cos(2\omega) = +0.894623651; -0.060353958; -0.834269693$$

$$2f\cos(2\omega) = (2f)(+0.894623651; -0.060353958; -0.834269693)$$
$$= (1.047144962)(+0.894623651; -0.060353958; -0.834269693)$$
$$= +0.936800649; -0.063199343; -0.873601306$$
$$= \ell$$

Now since,

$$u = \ell - V$$
$$= (+0.936800649; -0.063199343; -0.873601306) - (-1.063199347)$$
$$= +2, +1, +0.189598041$$
$$= R, S, T$$

Check,

$$R^3 - 3.189598041R^2 + 2.568794123R - 0.379196082 = 0$$
$$2^3 - 3.189598041(2)^2 + 2.568794123(2) - 0.379196082 = 0$$
$$8 - 12.75839216 + 5.137588246 - 0.379196082 = 0$$
$$0.379196082 - 0.379196082 = 0$$
$$0 = 0$$

$$S^3 - 3.189598041S^2 + 2.568794123S - 0.379196082 = 0$$
$$1^3 - 3.189598041(1)^2 + 2.568794123(1) - 0.379196082 = 0$$
$$1 - 3.189598041 + 2.568794123 - 0.379196082 = 0$$
$$0.379196082 - 0.379196082 = 0$$
$$0 = 0$$

$$T^3 - 3.189598041T^2 + 2.568794123T - 0.379196082 = 0$$
$$(0.189598041)^3 - 3.189598041(0.189598041)^2 + 2.568794123(0.189598041) - 0.379196082 = 0$$
$$0.006815559^3 - 0.114657811 + 0.487038333 - 0.379196082 = 0$$
$$0.379196082 - 0.379196082 = 0$$
$$0 = 0$$

PROBLEM NUMBER 27 *(Ref. Sections 13.3.3 and 13.3.4)*

GIVEN:

Three *3θ Cubic Equation* shown below:

$$z^3 - 3\zeta z^2 - 3z + \zeta = 0$$

Where,

$$\zeta = \sqrt{3}$$

DETERMINE:

The three roots via the *Cubic Resolution Transform*

SOLUTION:

Substituting $z = u$ into the *above given equation* renders:

$$u^3 - 3\zeta u^2 - 3u + \zeta = 0$$

Equation 42 is selected from the excerpt since it matches the above format,

$$u^3 + (3V)u^2 + 3(V^2 - f^2)u + V^3 - 3f^2V - 2\psi f^3 = 0 \quad \textit{[Ref. Equation 42]}$$

Comparing respective coefficients with those of the *given 3θ Cubic Equation* returns the following results,

$$V = -\zeta$$

$$\begin{aligned} f &= \sqrt{V^2 + 1} \\ &= \sqrt{\zeta^2 + 1} \\ &= \sqrt{(\sqrt{3})^2 + 1} \\ &= \sqrt{3 + 1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \psi &= \frac{V^3 - 3f^2V - \zeta}{2f^3} \\ &= \frac{-\zeta^3 + 3(2)^2\zeta - \zeta}{2(2)^3} \\ &= \frac{-3\zeta + 12\zeta - \zeta}{16} \\ &= \frac{8\zeta}{16} \\ &= \frac{\sqrt{3}}{2} \\ &= \cos(6\omega) \end{aligned}$$

Since $-1 \le \cos(6\omega) \le +1$ the *three roots* are *all real*. Hence,

At $\cos(6\omega) = \frac{\sqrt{3}}{2}$,

$$6\omega = \pm 30^o; +(30^o + 360^o)$$
$$= +30^o; +330^o; +390^o$$

$$2\omega = \frac{+30^o}{3}; \frac{+330^o}{3}; \frac{+390^o}{3}$$
$$= +10^o; 110^o; 130^o$$

$$\cos(2\omega) = +0.984807753; -0.342020143; -0.642787609$$

$$2f\cos(2\omega) = (2f)(0.984807753; -0.342020143; -0.642787609)$$
$$= 4(0.984807753; -0.342020143; -0.642787609)$$
$$= 3.939231012; -1.368080573; -2.571150439$$
$$= \ell$$

Now since,

$$u = \ell - V$$
$$= (3.939231012; -1.368080573; -2.571150439) - (-\zeta)$$
$$= (3.939231012; -1.368080573; -2.571150439) + (1.732050808)$$
$$= +5.67128182, +0.363970234, -0.839099631$$
$$= z_T; z_R; z_S$$

Q.E.D

Check,

$$z_R^{\ 3} - 3\zeta z_R^{\ 2} - 3z_R + \zeta = 0$$
$$(0.363970234)^3 - 3\sqrt{3}(0.363970234)^2 - 3(0.363970234) + \sqrt{3} = 0$$
$$0.048216713 - 0.688356818 - 1.091910703 + 1.732050808 = 0$$
$$-1.732050808 + 1.732050808 = 0$$
$$0 = 0$$

$$z_S^{\ 3} - 3\zeta z_S^{\ 2} - 3z_S + \zeta = 0$$
$$(-0.839099631)^3 - 3\sqrt{3}(-0.839099631)^2 - 3(-0.839099631) + \sqrt{3} = 0$$
$$-0.590800141 - 3.658549561 + 2.517298894 + 1.732050808 = 0$$
$$-1.732050808 + 1.732050808 = 0$$
$$0 = 0$$

$$z_T^{\ 3} - 3\zeta z_T^{\ 2} - 3z_T + \zeta = 0$$
$$(5.67128182)^3 - 3\sqrt{3}(5.67128182)^2 - 3(5.67128182) + \sqrt{3} = 0$$
$$182.4079182 - 167.1261236 - 17.01384546 + 1.732050808 = 0$$
$$-1.732050808 + 1.732050808 = 0$$
$$0 = 0$$

PROBLEM NUMBER 28 *(Ref. Sections 13.3.3 and 13.3.4)*

GIVEN:

$$f^3 - 25.621f^2 + 78.223 = 0$$

RESOLVE:

The above given equation

SOLUTION:

By applying the *predominant form of the CRT:*

$$f^3 \pm \left(\frac{3\ell}{2\psi}\right)f^2 \mp \left(\frac{\ell^3}{2\psi}\right) = 0 \qquad \text{[Ref. Equation 38]}$$

First, matching *second term coefficients gives:*

$$-25.621 = \frac{3\ell}{2\psi}$$

$$2\psi = -\frac{3\ell}{25.621}$$

Then, matching *third term coefficients* renders:

$$78.223 = -\frac{\ell^3}{2\psi}$$

Substitution for 2ψ into this result yields:

$$78.223 = -\frac{\ell^3}{\left(3\ell/25.621\right)}$$

Such that

$$\ell = \sqrt{3\left(78.223/25.621\right)} = 3.026424354$$

Whereby,

$$\psi = \cos(6\theta) = -\frac{3\ell}{2(25.621)} - \frac{3(3.026424354)}{2(25.621)} = -0.1771842056$$

Since $-1 \le \cos(6\omega) \le +1$ the *three roots* are *all real*. Hence,

$$6\omega = +100.2057905^o; -100.2057905^o; +(100.2057905^o + 360^o)$$
$$= +100.2057905^o; +259.7942095^o; +460.2057905^o$$

$$2\omega = \frac{+100.2057905^o}{3}; \frac{+259.7942095^o}{3}; \frac{+460.2057905^o}{3}$$
$$= +33.40193018^o; +86.59805982^o; +153.4019302^o$$

$$\cos(2\omega) = +0.834829318, +0.059340002, -0.89416932$$

Therefore,

$$2f\cos(2\omega) = \ell$$

$$f = \frac{\ell}{2\cos(2\omega)}$$

$$= \frac{3.026424354}{2(0.834829318)};\frac{3.026424354}{2(0.059340002)};\frac{3.026424354}{2(-0.89416932)}$$

$$= 1.812600665; 25.50070991; -1.692310554$$

Check:

$$f^3 - 25.621f^2 + 78.223 = 0$$

$$1.812600654^3 - 25.621(1.812600654)^2 + 78.223 = 0$$
$$5.955337852 - 84.17833785 + 78.223 = 0$$
$$78.223 + 78.223 = 0$$
$$0 = 0$$

$$25.50070989^3 - 25.621(25.50070989)^2 + 78.223 = 0$$
$$16582.75985 - 16660.98285 + 78.223 = 0$$
$$78.223 + 78.223 = 0$$
$$0 = 0$$

$$-1.692310553^3 - 25.621(-1.692310553)^2 + 78.223 = 0$$
$$-4.84663359 - 73.37636641 + 78.223 = 0$$
$$78.223 + 78.223 = 0$$
$$0 = 0$$

PROBLEM NUMBER 29 *(Ref. Sections 13.3.3 and 13.3.4)*

GIVEN:

$$\ell^3 - \ell + \frac{2\sqrt{3}}{9} = 0$$

RESOLVE:

The *above given equation*

SOLUTION:

By applying the *re-arranged form of* the *RCT*:

$$\ell^3 - \left(3f^2\right)\ell - 2\psi\left(f^3\right) = 0 \quad \text{[Ref. Equation 41]}$$

First, matching ℓ *term coefficients* gives:

$$3f^2 = +1$$

Or

$$f = \frac{\sqrt{3}}{3}$$

Matching and substitution into the *third term coefficient* of the *above given equation* yields:

$$2\psi\left(f^3\right) = -\frac{2\sqrt{3}}{9}$$

$$\psi = \frac{-\frac{2\sqrt{3}}{9}}{2f^3} = \frac{-\frac{2\sqrt{3}}{9}}{2(\frac{3\sqrt{3}}{27})} = -1 = \cos(6\omega)$$

Since $-1 \le \cos(6\omega) \le +1$ the *three roots* are *all real*. Hence,

$$6\omega = 180^o;-180^o;+(180^o + 360^o)$$

$$= +180^o;+180^o;+540^o$$

$$2\omega = \frac{+180^o}{3};\frac{+180^o}{3};\frac{+540^o}{3}$$

$$= +60^o;+60^o;+180^o$$

$$\cos(2\omega) = +0.5;+0.5;-1$$

Therefore,

$$\ell = 2f\cos(2\omega)$$

$$= 2\frac{\sqrt{3}}{3}(0.5;0.5;-1)$$

$$= \frac{\sqrt{3}}{3};\frac{\sqrt{3}}{3};-\frac{2\sqrt{3}}{3}$$

Check:

$$\ell^3 - \ell + \frac{2\sqrt{3}}{9} = 0$$

$$(\frac{\sqrt{3}}{3})^3 - \frac{\sqrt{3}}{3} + \frac{2\sqrt{3}}{9} = 0$$

$$\frac{3\sqrt{3}}{3(9)}) - \frac{\sqrt{3}}{3}(\frac{3}{3}) + \frac{2\sqrt{3}}{9} = 0$$

$$\frac{\sqrt{3}}{9}(1 - 3 + 2) = 0$$

$$\frac{\sqrt{3}}{9}(0) = 0$$

$$0 = 0$$

$$(-\frac{2\sqrt{3}}{3})^3 - (-\frac{2\sqrt{3}}{3}) + \frac{2\sqrt{3}}{9} = 0$$

$$-\frac{(8)3\sqrt{3}}{3(9)}) + \frac{2\sqrt{3}}{3}(\frac{3}{3}) + \frac{2\sqrt{3}}{9} = 0$$

$$\frac{\sqrt{3}}{9}(-8 + 6 + 2) = 0$$

$$\frac{\sqrt{3}}{9}(0) = 0$$

$$0 = 0$$

Lastly,

$$(\ell - \ell_R)(\ell - \ell_S)(\ell - \ell_T) = 0$$

$$(\ell - \ell_R)[\ell^2 - (\ell_S + \ell_T)\ell + \ell_S \ell_T] = 0$$

$$\ell^3 - (\ell_R + \ell_S + \ell_T)\ell^2 + [\ell_R(\ell_S + \ell_T) + \ell_S \ell_T]\ell - \ell_R \ell_S \ell_T = 0$$

$$\ell^3 - (\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} - \frac{2\sqrt{3}}{3})\ell^2 + [\frac{\sqrt{3}}{3}(\frac{\sqrt{3}}{3} - \frac{2\sqrt{3}}{3}) - \frac{\sqrt{3}}{3}(\frac{2\sqrt{3}}{3})]\ell + \frac{\sqrt{3}}{3}(\frac{\sqrt{3}}{3})(\frac{2\sqrt{3}}{3}) = 0$$

$$\ell^3 - (0)\ell^2 + [\frac{\sqrt{3}}{3}(-\frac{\sqrt{3}}{3}) - \frac{\sqrt{3}}{3}(\frac{2\sqrt{3}}{3})]\ell + (\frac{1}{3})(\frac{2\sqrt{3}}{3}) = 0$$

$$\ell^3 + \frac{\sqrt{3}}{3}(-\frac{\sqrt{3}}{3} - \frac{2\sqrt{3}}{3})\ell + \frac{2\sqrt{3}}{9}) = 0$$

$$\ell^3 + \frac{\sqrt{3}}{3}(-\frac{3\sqrt{3}}{3})\ell + \frac{2\sqrt{3}}{9}) = 0$$

$$\ell^3 - \ell + \frac{2\sqrt{3}}{9} = 0$$

PROBLEM NUMBER 30 *(Ref. Sections 13.3.3 and 13.3.5)*

GIVEN:

$x^3 + 6x - 20 = 0$

RESOLVE:

The *above given equation*

SOLUTION:

Substituting $x = \ell$ into the *above given equation* renders:

$$\ell^3 + 6\ell - 20 = 0$$

By applying the *re-arranged form of* the *RCT*:

$$\ell^3 - \left(3f^2\right)\ell - 2\psi\left(f^3\right) = 0$$ *[Ref. Equation 41]*

First, matching *second term coefficients* gives:

$$3f^2 = -6$$

Or

$$f = \sqrt{2}i$$

Matching *third term coefficients* realizes: follows:

$$2\psi\left(f^3\right) = 20$$

$$\psi = \frac{20}{2f^3} = \frac{20}{2(2\sqrt{2})i^3}\left(\frac{i}{i}\right) = \frac{5\sqrt{2}}{2}i = 3.535533906i = \cos(6\omega)$$

Considering that,

$$\cos(2\omega) \pm i\sin(2\omega) = \sqrt[3]{\cos(6\omega) \pm i\sin(6\omega)}$$ *[Ref. Section 21]*

Where,

$$\ell = 2f\cos(2\omega)$$

$$= f\left[\cos(2\omega) + i\sin\ (2\omega) + \cos(2\omega) - i\sin(2\omega)\right]$$

$$= f\left[\sqrt[3]{\cos(6\omega) + i\sin(6\omega)} + \sqrt[3]{\cos(6\omega) - i\sin(6\omega)}\right]$$

And where,

$$i\sin(6\omega) = i\sqrt{1 - \cos^2(6\omega)}$$

$$= \sqrt{-1}\sqrt{1 - \psi^2}$$

$$= \sqrt{(-1)(1 - \psi^2)}$$

$$= \sqrt{\psi^2 - 1}$$

By substitution,

$$\ell = f\left[\sqrt[3]{\psi+\sqrt{\psi^2-1}}+\sqrt[3]{\psi-\sqrt{\psi^2-1}}\right]$$

$$= i\sqrt{2}\left[\sqrt[3]{3.535533906i+\sqrt{(3.535533906i)^2-1}}+\sqrt[3]{3.535533906i-\sqrt{(3.535533906i)^2-1}}\right]$$

$$= i\sqrt{2}\left[\sqrt[3]{3.535533906i+\sqrt{12.5i^2-1}}+\sqrt[3]{3.535533906i-\sqrt{12.5i^2-1}}\right]$$

$$= i\sqrt{2}\left[\sqrt[3]{3.535533906i+\sqrt{-13.5}}+\sqrt[3]{3.535533906i-\sqrt{-13.5}}\right]$$

$$= i\sqrt{2}\left[\sqrt[3]{3.535533906i+3.674234614i}+\sqrt[3]{3.535533906i-3.674234614i}\right]$$

$$= i\sqrt{2}\left[\sqrt[3]{7.20976852i}+\sqrt[3]{-.1387007082i}\right]$$

$$= i\sqrt{2}[-1.931851653i+0.5176380902i]$$

$$= i\sqrt{2}[-1.414213562i]$$

$$= +2$$

$$= x$$

The *complex roots* are determined as follows:

For,

$$x_R = 2$$

$$x_R - 2 = 0$$

$$(x-2)(x^2+Mx+N) = x^3+6x-20$$

$$x^3+(M-2)x^2+(N-2M)x-2N = x^3+6x-20$$

Comparing like *coefficients* gives:

$$M-2=0$$
$$M=2$$

$$2N=20$$
$$N=10$$

$$N-2M = 10-2(2)$$
$$= +6$$

Therefore,

$$x^2+Mx+N=0$$
$$x^2+2x+10=0$$

$$ax^2+bx+c=0$$

Via Quadratic Formula:

$$x_S; x_T = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$$
$$= \frac{1}{2(1)}[-2 \pm \sqrt{2^2 - 4(1)(10)}]$$
$$= \frac{1}{2}(-2 \pm \sqrt{4 - 40})$$
$$= \frac{1}{2}(-2 \pm \sqrt{-36})$$
$$= \frac{-2 \pm 6i}{2}$$
$$= -1 \pm 3i$$

Check,

$$(x - x_R)(x - x_S)(x - x_T) = 0$$
$$(x - x_R)[x^2 - (x_S + x_T)x + x_S x_T] = 0$$
$$x^3 - (x_R + x_S + x_T)x^2 + [x_R(x_S + x_T) + x_S x_T]x - x_R x_S x_T = 0$$
$$x^3 - (2 - 1 + 3i - 1 - 3i)x^2 + [2(-1 + 3i - 1 - 3i) + (-1 + 3i)(-1 - 3i)]x - 2(-1 + 3i)(-1 - 3i) = 0$$
$$x^3 - (0)x^2 + [2(-2) + (1 - 9i^2)]x - 2(1 - 9i^2) = 0$$
$$x^3 + (-4 + 1 + 9)x - 2(1 + 9) = 0$$
$$x^3 + 6x - 20 = 0$$

PROBLEM NUMBER 31 (Ref. Sections 13.3.3 and 13.3.5)

GIVEN:

$$z^3 + 3z^2 - 2z - \frac{2431}{64} = 0$$

DETERMINE:

The three roots via the *Cubic Resolution Transform*

SOLUTION:

Substituting $z = u$ into the *above given equation* renders:

$$u^3 + 3u^2 - 2u - \frac{2431}{64} = 0$$

Equation 42 is selected from the excerpt since it matches the above format,

$$u^3 + (3V)u^2 + 3(V^2 - f^2)u + V^3 - 3f^2V - 2\psi f^3 = 0 \quad \textit{[Ref. Equation 42]}$$

Comparing respective coefficients with those of the *given 3θ Cubic Equation* returns the following results,

$$V = 1$$

$$\begin{aligned} f &= \sqrt{V^2 + 2/3} \\ &= \sqrt{1 + 2/3} \\ &= \sqrt{\frac{5}{3}} \\ &= 1.290994449 \end{aligned}$$

$$\psi = \frac{V^3 - 3f^2V + 2431/64}{2f^3}$$

$$\begin{aligned} \cos(6\omega) &= \frac{(1)^3 - 3(\frac{\sqrt{15}}{3})^2(1) + \frac{2431}{64}}{2(\frac{\sqrt{15}}{3})^3} \\ &= \frac{27[2431 - 4(64)]\sqrt{15}}{30(64)15} \\ &= \frac{9(2175)\sqrt{15}}{10(64)15} \\ &= \frac{9(435)}{30(64)}\sqrt{15} \\ &= \frac{3(435)}{640}\sqrt{15} \\ &= 7.897255104 \end{aligned}$$

Since $\cos(6\omega) > +1$, two of the roots mist be imaginary.

Considering that,

$$\cos(2\omega) \pm i\sin(2\omega) = \sqrt[3]{\cos(6\omega) \pm i\sin(6\omega)} \qquad \textit{[Ref. Section 21]}$$

Where,

$$\ell = 2f\cos(2\omega)$$
$$= f\left[\cos(2\omega) + i\sin\ (2\omega) + \cos(2\omega) - i\sin(2\omega)\right]$$
$$= f\left[\sqrt[3]{\cos(6\omega) + i\sin(6\omega)} + \sqrt[3]{\cos(6\omega) - i\sin(6\omega)}\right]$$

And where,

$$i\sin(6\omega) = i\sqrt{1 - \cos^2(6\omega)}$$
$$= \sqrt{-1}\sqrt{1 - \psi^2}$$
$$= \sqrt{(-1)(1 - \psi^2)}$$
$$= \sqrt{\psi^2 - 1}$$

By substitution,

$$\ell = f\left[\sqrt[3]{\psi + \sqrt{\psi^2 - 1}} + \sqrt[3]{\psi - \sqrt{\psi^2 - 1}}\right]$$
$$= \sqrt{\frac{5}{3}}\left[\sqrt[3]{7.897255104 + \sqrt{(7.897255104)^2 - 1}} + \sqrt[3]{7.897255104 - \sqrt{(7.897255104)^2 - 1}}\right]$$
$$= \sqrt{\frac{5}{3}}\left[\sqrt[3]{7.897255104 + \sqrt{62.36663818 - 1}} + \sqrt[3]{7.897255104 - \sqrt{62.36663818 - 1}}\right]$$
$$= \sqrt{\frac{5}{3}}\left[\sqrt[3]{7.897255104 + \sqrt{61.36663818}} + \sqrt[3]{7.897255104 - \sqrt{61.36663818}}\right]$$
$$= \sqrt{\frac{5}{3}}\left[\sqrt[3]{7.897255104 + 7.833686117} + \sqrt[3]{7.897255104 - 7.833686117}\right]$$
$$= \sqrt{\frac{5}{3}}\left[\sqrt[3]{15.73094122} + \sqrt[3]{0.0635689871}\right]$$
$$= \sqrt{\frac{5}{3}}[2.505637476 + 0.3991000332]$$
$$= \sqrt{\frac{5}{3}}[2.90473751]$$
$$= 3.75$$

Now since,

$$\begin{aligned} u &= \ell - V \\ &= 3.75 - 1 \\ &= 2.75 \\ &= \frac{11}{4} \\ &= z_R \end{aligned}$$

Check,

$$\begin{aligned} z_R{}^3 + 3z_R{}^2 - 2z_R - \frac{2431}{64} &= \left(\frac{11}{4}\right)^3 + 3\left(\frac{11}{4}\right)^2 - 2\left(\frac{11}{4}\right) - \frac{2431}{64} \\ &= \frac{1331}{64} + \frac{1452}{64} - \frac{352}{64} - \frac{2431}{64} \\ &= \frac{2783}{64} - \frac{2783}{64} \\ &= 0 \end{aligned}$$

Lastly, the *imaginary roots* are determined as follows:

For,

$$\begin{aligned} z_R &= 2.75 \\ z_R - \frac{11}{4} &= 0 \end{aligned}$$

$$\left(z - \frac{11}{4}\right)(z^2 + Mz + N) = z^3 + 3z^2 - 2z - \frac{2431}{64}$$

$$z^3 + (M - \frac{11}{4})z^2 + (N - \frac{11}{4}M)z - \frac{11}{4}N = z^3 + 3z^2 - 2z - \frac{2431}{64}$$

Comparing like *coefficients* gives:

$$M - \frac{11}{4} = 3$$

$$M = \frac{23}{4}$$

$$\frac{11}{4}N = \frac{2431}{64}$$

$$N = \frac{2431}{16(11)}$$

$$\begin{aligned} N - \frac{11}{4}M &= \frac{2431}{16(11)} - \frac{11}{4}(\frac{23}{4}) \\ &= \frac{221 - 253}{16} \\ &= -2 \end{aligned}$$

Therefore,

$$x^2 + Mx + N = 0$$

$$x^2 + \frac{23}{4}x + \frac{221}{16} = 0$$

$$ax^2 + bx + c = 0$$

Via Quadratic Formula:

$$z_S ; z_T = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$$

$$= \frac{1}{2(1)}[-\frac{23}{4} \pm \sqrt{(\frac{23}{4})^2 - 4(1)(\frac{221}{16})}]$$

$$= \frac{1}{8}(-23 \pm \sqrt{529 - 884})$$

$$= \frac{1}{8}(-23 \pm \sqrt{-355})$$

$$= \frac{-23 \pm 18.84144368i}{8}$$

$$= -2.875 \pm 2.35518046i$$

Check,

$$(z - z_R)(z - z_S)(z - z_T) = 0$$

$$(z - z_R)[z^2 - (z_S + z_T)z + z_S z_T] = 0$$

$$z^3 - (z_R + z_S + z_T)z^2 + [z_R(z_S + z_T) + z_S z_T]z - z_R z_S z_T = 0$$

For the *Generalized Cubic Equation* written below:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \textit{[Ref. Equation 32]}$$

$$\beta = -(z_R + z_S + z_T)$$

$$= -(2.75 - 2.875 + 2.35518046i - 2.875 - 2.35518046i)$$

$$= -(2.75 - 5.75)$$

$$= 3$$

$$\gamma = z_R(z_S + z_T) + z_S z_T$$

$$= 2.75(-2.875 + 2.35518046i - 2.875 - 2.35518046i) + (-2.875 + 2.35518046i)(-2.875 - 2.35518046i)$$

$$= 2.75(-5.75) + (-2.875)^2 - (i)^2(2.355180646)^2$$

$$= -15.8125 + 8.265625 + 5.546875875$$

$$= -2$$

$$\delta = -z_R z_S z_T$$
$$= -(2.75)(-2.875 + 2.35518046i)(-2.875 - 2.35518046i)$$
$$= -(2.75)[(-2.875)^2 - (i)^2(2.355180646)^2]$$
$$= -2.75(8.265625 + 5.546875875)$$
$$= -2.75(13.8125)$$
$$= -\frac{2431}{64}$$

$$z_S{}^3 + 3z_S{}^2 - 2z_S - \frac{2431}{64} = 0$$
$$(-2.875 + 2.35518046i)^3 + 3(-2.875 + 2.35518046i)^2 - 2(-2.875 + 2.35518046i) - \frac{2431}{64} = 0$$
$$(24.078125 + 45.33722386i) + (8.15625 - 40.62686294i) + (5.75 - 4.71036092i) - \frac{2431}{64} = 0$$
$$37.984375 + 45.33722386i - 45.33722386 - 37.984375 = 0$$
$$0 + 0i = 0$$

$$z_T{}^3 + 3z_T{}^2 - 2z_T - \frac{2431}{64} = 0$$
$$(-2.875 - 2.35518046i)^3 + 3(-2.875 - 2.35518046i)^2 - 2(-2.875 - 2.35518046i) - \frac{2431}{64} = 0$$
$$(24.078125 - 45.33722386i) + (8.15625 + 40.62686294i) + (5.75 + 4.71036092i) - \frac{2431}{64} = 0$$
$$37.984375 - 45.33722386i + 45.33722386 - 37.984375 = 0$$
$$0 + 0i = 0$$

PROBLEM NUMBER 32 *(Ref. Sections 13.3.3 and 13.3.5)*

GIVEN:

$$z^3 + \left(\frac{1}{4}\right)z^2 - \frac{3}{16} = 0$$

RESOLVE:

The above given equation

SOLUTION:

Substituting $z = f$ into the *above given equation* renders:

$$f^3 + \left(\frac{1}{4}\right)f^2 - \frac{3}{16} = 0$$

By applying the *predominant form of the CRT:*

$$f^3 \pm \left(\frac{3\ell}{2\psi}\right)f^2 \mp \left(\frac{\ell^3}{2\psi}\right) = 0 \qquad \text{[Ref. Equation 38]}$$

First, matching *second term coefficients gives:*

$$\frac{1}{4} = \frac{3\ell}{2\psi}$$

$$2\psi = 12\ell$$

Then, matching *third term coefficients* renders:

$$\frac{3}{16} = -\frac{\ell^3}{2\psi}$$

Substitution for 2ψ into this result yields:

$$\frac{3}{16} = -\frac{\ell^3}{12\ell}$$

Such that

$$\ell = \sqrt{\frac{36}{16}} = \frac{6}{4} = 1.5$$

Whereby,

$$\psi = \cos(6\omega) = 6\ell = 6(1.5) = 9$$

Since $\cos(6\omega) > +1$ two *roots* must be *imaginary*. Hence,

$$\psi = \cos(6\omega) = \cosh[i(6\omega)] = 9$$

$$i(6\omega) = 2.88727095$$

$$6\omega = -2.88727095i$$

$$2\omega = -.0.96242365i$$

$$\cos(2\omega) = \cos(-.0.96242365i) = \cosh(-.0.96242365) = 1.5$$

Therefore,

$$2f\cos(2\omega) = \ell$$

$$f = \frac{\ell}{2\cos(2\omega)}$$

$$= \frac{1.5}{2(1.5)}$$

$$= \frac{1}{2}$$

$$= z_R$$

Check:

$$z_R^{\ 3} + \left(\frac{1}{4}\right) z_R^{\ 2} - \frac{3}{16} = 0$$

$$(\frac{1}{2})^3 + \left(\frac{1}{4}\right)(\frac{1}{2})^2 - \frac{3}{16} = 0$$

$$(\frac{1}{8})(\frac{2}{2}) + \left(\frac{1}{16}\right) - \frac{3}{16} = 0$$

$$\frac{3}{16} - \frac{3}{16} = 0$$

$$0 = 0$$

Verification:

Considering that,

$$\cos(2\omega) \pm i\sin(2\omega) = \sqrt[3]{\cos(6\omega) \pm i\sin(6\omega)} \qquad \text{[Ref. Section 21]}$$

Where,

$$\ell = 2f\cos(2\omega)$$

$$= f\left[\cos(2\omega) + i\sin\ (2\omega) + \cos(2\omega) - i\sin(2\omega)\right]$$

$$= f\left[\sqrt[3]{\cos(6\omega) + i\sin(6\omega)} + \sqrt[3]{\cos(6\omega) - i\sin(6\omega)}\right]$$

And where,

$$i\sin(6\omega) = i\sqrt{1-\cos^2(6\omega)}$$

$$= \sqrt{-1}\sqrt{1-\psi^2}$$

$$= \sqrt{(-1)(1-\psi^2)}$$

$$= \sqrt{\psi^2 - 1}$$

By substitution,

$$\ell = f\left[\sqrt[3]{\psi + \sqrt{\psi^2 - 1}} + \sqrt[3]{\psi - \sqrt{\psi^2 - 1}}\right]$$

$$= \tfrac{1}{2}\left[\sqrt[3]{9 + \sqrt{9^2 - 1}} + \sqrt[3]{9 - \sqrt{9^2 - 1}}\right]$$

$$= \tfrac{1}{2}\left[\sqrt[3]{9 + \sqrt{81 - 1}} + \sqrt[3]{9 - \sqrt{81 \ - 1}}\right]$$

$$= \tfrac{1}{2}\left[\sqrt[3]{9 + \sqrt{80}} + \sqrt[3]{9 - \sqrt{80}}\right]$$

$$= \tfrac{1}{2}\left[\sqrt[3]{9 + 8.94427191} + \sqrt[3]{9 - 8.94427191}\right]$$

$$= \tfrac{1}{2}\left[\sqrt[3]{17.94427191} + \sqrt[3]{0.05572809}\right]$$

$$= \tfrac{1}{2}\left[2.618033989 + 0.3819660113\right]$$

$$= \tfrac{1}{2}[3]$$

$$= 1.5$$

The *imaginary roots* are determined as follows, where:

$$z_R = \frac{1}{2}$$

$$z_R - \frac{1}{2} = 0$$

$$(z - \frac{1}{2})(z^2 + Mz + N) = z^3 + \left(\frac{1}{4}\right)z^2 - \frac{3}{16}$$

$$z^3 + (M - \frac{1}{2})z^2 + (N - \frac{1}{2}M)z - \frac{1}{2}N = z^3 + \left(\frac{1}{4}\right)z^2 - \frac{3}{16}$$

Comparing like *coefficients* gives:

$$M - \frac{1}{2} = \frac{1}{4}$$

$$M = \frac{3}{4}$$

$$\frac{1}{2}N = \frac{3}{16}$$

$$N = \frac{3}{8}$$

$$N - \frac{1}{2}M = \frac{3}{8} - \frac{1}{2}(\frac{3}{8})$$

$$= -\frac{3}{16}$$

Therefore,

$$x^2 + Mx + N = 0$$

$$x^2 + \frac{3}{4}x + \frac{3}{8} = 0$$

$$ax^2 + bx + c = 0$$

Via Quadratic Formula:

$$z_S ; z_T = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$$

$$= \frac{1}{2(1)}[-\frac{3}{4} \pm \sqrt{(\frac{3}{4})^2 - 4(1)(\frac{3}{8})(\frac{2}{2})}]$$

$$= \frac{1}{8}(-3 \pm \sqrt{9 - 24})$$

$$= \frac{1}{8}(-3 \pm \sqrt{-15})$$

$$= \frac{-3 \pm 3.872983346i}{8}$$

$$= -0.375 \pm 0.484122918i$$

Check,

$$(z - z_R)(z - z_S)(z - z_T) = 0$$

$$(z - z_R)[z^2 - (z_S + z_T)z + z_S z_T] = 0$$

$$z^3 - (z_R + z_S + z_T)z^2 + [z_R(z_S + z_T) + z_S z_T]z - z_R z_S z_T = 0$$

For the *Generalized Cubic Equation* written below:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \textit{[Ref. Equation 32]}$$

$$\beta = -(z_R + z_S + z_T)$$

$$= -(0.5 - 0.375 + 0.484122918i - 0.375 - 0.484122918i)$$

$$= -(0.5 - 0.75)$$

$$= +0.25$$

$$\gamma = z_R(z_S + z_T) + z_S z_T$$

$$= 0.5(-0.375 + 0.484122918i - 0.375 - 0.484122918i) + (-0.375 + 0.484122918i)(-0.375 - 0.484122918i)$$

$$= 0.5(-0.75) + (-0.375)^2 - (i)^2(0.484122918)^2$$

$$= -0.375 + 0.14025 + 0.234375$$

$$= 0$$

$$\delta = -z_R z_S z_T$$
$$= -(0.5)(-0.375 + 0.484122918i)(-0.375 - 0.484122918i)$$
$$= -(0.5)[(-0.375)^2 - (i)^2 (0.484122918)^2]$$
$$= -0.5(0.14025 + 0.234375)$$
$$= -0.5(0.375)$$
$$= -\frac{1}{2}(\frac{3}{8})$$
$$= -\frac{3}{16}$$

$$z_{S;T}{}^3 + \left(\frac{1}{4}\right) z_{S;T}{}^2 - \frac{3}{16} = 0$$
$$(-0.375 \pm 0.484122918i)^3 + 0.25(-0.375 \pm 0.484122918i)^2 - \frac{3}{16} = 0$$
$$(0.210937499 \pm 0.090773047i) - (0.023437499 \pm 0.090773047i) - \frac{3}{16} = 0$$
$$0.1875 - 0.1875 = 0$$
$$0 = 0$$

PROBLEM NUMBER 33 *(Ref. Sections 13.3.3 and 13.3.5)*

GIVEN:

RESOLVE:

The above given equation

SOLUTION:

Substituting $x = u$ into the *above given equation* renders:

$$u^3 - \left(\frac{1}{4}\right)u^2 - \frac{40}{216} = 0$$

By applying the *transformed intermediate form of the CRT:*

$$u^3 - \left(\frac{3\ell}{2\psi}\right)u^2 + \frac{\ell^3}{2\psi^3}\left(1 - \psi^2\right) = 0 \qquad \textit{[Ref. Equation 40]}$$

First, matching *second term coefficients gives:*

$$\frac{1}{4} = \frac{3\ell}{2\psi}$$

$$2\psi = 12\ell$$

$$\psi = 6\ell$$

Then, matching *third term coefficients* renders:

$$-\frac{40}{216} = \frac{\ell^3}{2\psi^3}(1 - \psi^2)$$

$$-\frac{40}{216} = \frac{\ell^3}{2(6\ell)^3}[1 - (6\ell)^2]$$

$$-\frac{40}{108} = \frac{1 - 36\ell^2}{(6)(36)}$$

$$-\frac{40(36)}{18} = 1 - 36\ell^2$$

$$-80 - 1 = -36\ell^2$$

$$\frac{81}{36} = \ell^2$$

$$\sqrt{\frac{81}{36}} = \ell$$

$$1.5 = \ell$$

$$\psi = 6\ell$$
$$= 6(1.5)$$
$$= 9$$

Since the above values of $\ell = 1.5$ and $\psi = 9$ are identical to those determined in the *last problem*, an *association* is drawn between the respective two *given equations* as follows:

Whereas *Equation 40* shown above is a derivation of the *predominant form of the CRT (Ref. Equation 38)*, and is obtained by letting $f = u + V$ such that,

$$V = \frac{-\ell}{\psi}$$

Then,

$$f = u + V$$
$$= u - \frac{\ell}{\psi}$$
$$= u - \frac{3/2}{9}$$
$$= u - \frac{1}{6}$$
$$f + \frac{1}{6} = u$$

Substitution into the *transformation* of the *given equation* above renders:

$$u^3 - \left(\frac{1}{4}\right)u^2 - \frac{40}{216} = 0$$
$$(f + \frac{1}{6})^3 - \left(\frac{1}{4}\right)(f + \frac{1}{6})^2 - \frac{40}{216} = 0$$

Expanding above terms yields:

$$[f^3 + 3(\frac{1}{6})f^2 + 3(\frac{1}{6})^2 f + (\frac{1}{6})^3] - (\frac{1}{4})[f^2 + 2(\frac{1}{6})f + (\frac{1}{6})^2] - \frac{40}{216} = 0$$
$$f^3 + f^2[3(\frac{1}{6})(\frac{2}{2}) - (\frac{1}{4})] + f[3(\frac{1}{6})^2(\frac{2}{2}) - \frac{2}{4}(\frac{1}{6})] + [(\frac{1}{6})^3(\frac{4}{4}) - (\frac{1}{4})(\frac{1}{6})^2(\frac{6}{6}) - \frac{40}{216}(\frac{4}{4})] = 0$$
$$f^3 + f^2(\frac{2-1}{4}) + f(\frac{2-2}{24}) + \frac{4-6-160}{4(216)} = 0$$
$$f^3 + (\frac{1}{4})f^2 + 0f - \frac{3}{4(4)} = 0$$

$$f^3 + (\frac{1}{4})f^2 - \frac{162}{4(216)} = 0$$

$$f^3 + (\frac{1}{4})f^2 - \frac{3(54)}{4(4)(54)} = 0$$

$$f^3 + (\frac{1}{4})f^2 - \frac{3}{16} = 0$$

Substituting $f = z$ into the above equation renders:

$$z^3 + \left(\frac{1}{4}\right)z^2 - \frac{3}{16} = 0$$

This turns out to be *identical* to the *given equation* presented in the *last problem* whose roots are:

$$z_{R;}z_S;z_T = \frac{1}{2};-0.375+0.484122918i;-0.375-0.484122918i$$

Then, the solution for this problem is determined as:

$$u = f + \frac{1}{6}$$

$$= z + \frac{1}{6}$$

$$= (\frac{1}{2};-\frac{3}{8}+0.484122918i;-\frac{3}{8}-0.484122918i)+(\frac{1}{6})$$

$$= \frac{2}{3};-\frac{5}{24}+0.484122918i;-\frac{5}{24}-0.484122918i$$

$$= u_{R;}u_S;u_T$$

$$= x_{R;}x_S;x_T$$

Check,

$$x_R{}^3 - \left(\frac{1}{4}\right)x_R{}^2 - \frac{40}{216} = 0$$

$$(\frac{2}{3})^3 - (\frac{1}{4})(\frac{2}{3})^2 - \frac{40}{216} = 0$$

$$(\frac{8}{27})(\frac{8}{8}) - (\frac{4}{36})(\frac{6}{6}) - \frac{40}{216} = 0$$

$$\frac{64-24-40}{216} = 0$$

$$\frac{40-40}{216} = 0$$

$$0 = 0$$

$$x_{S;T}{}^{3}-\left(\frac{1}{4}\right)x_{S;T}{}^{2}-\frac{40}{216}=0$$

$$(-\frac{5}{24}\pm 0.484122918i)^{3}-(\frac{1}{4})(-\frac{5}{24}\pm 0.484122918i)^{2}-\frac{40}{216}=0$$

$$(0.137442129\mp 0.05042947i)+(0.047743055\pm 0.05042947i)-0.185185185=0$$

$$0.137442129+0.047743055-0.185185185=0$$

$$0.185185185-0.185185185=0$$

$$0=0$$

PROBLEM NUMBER 34 *(Section 14.1)*

GIVEN:

$$\zeta = \sqrt{3}$$
$$R = 1$$
$$S = 2$$

DETERMINE:

The location where the *slope* of the *associated Parabolic Curve* is zero

SOLUTION:

For,

$$\zeta = \sqrt{3} = \tan(3\theta) = \tan 60^o$$
$$3\theta = 60^o$$
$$\theta = 20^o = \theta_R$$

Where,

$$\zeta(C+3D)\tan^2\theta - (B-3D)\tan\theta - \zeta(D+1) = 0 \qquad \text{[Ref. Equation 30]}$$

Being of the form:

$$a\tan^2\theta + b\tan\theta + c = 0$$

And,

$$B = -(R+S+T)$$
$$C = RS + RT + ST$$
$$D = -RST$$

Comparing like coefficients renders,

$$\begin{aligned} a = \zeta(C+3D) &= \zeta[(RS+RT+ST) - 3RST] \\ &= \zeta[RS + (R+S)T - 3RST] \\ &= \zeta[(1)(2) + (1+2)T - 3(1)(2)T] \\ &= \zeta[(2+3T) - 6T] \\ &= \zeta(2-3T) \end{aligned}$$

$$\begin{aligned} b = -(B-3D) &= R + S + T - 3RST \\ &= (1+2+T) - 3(1)(2)T \\ &= 3 - 5T \end{aligned}$$

$$\begin{aligned} c = \zeta(D+1) &= \zeta(RST - 1) \\ &= \zeta[(1)(2)T - 1] \\ &= \zeta(2T-1) \end{aligned}$$

Then,

$$\zeta(2-3T)x^2+(3-5T)x+\zeta(2T-1)=0$$
$$\sqrt{3}(2-3T)x^2+(3-5T)x+\sqrt{3}(2T-1)=0$$

The value of T is determined as follows:

$$z_S = S\tan\theta = 2\tan\theta = 2\tan 20^o = \tan\theta_S = 0.727940468$$
$$\theta_S = \arctan 0.727940468$$
$$= 36.05238873^o$$

For $3\theta = 60^o$

$$\theta_R+\theta_S+\theta_T = 60^o$$
$$\theta_T = 60^o-(\theta_R+\theta_S)$$
$$= 60^o-(20^o+36.05238873^o)$$
$$= 3.947611267^o$$

$$\tan\theta_T = T\tan\theta = \tan 3.947611267^o$$
$$T = \frac{\tan 3.947611267^o}{\tan 20^o}$$
$$= \frac{0.069008043}{0.363970234}$$
$$= 0.189598041$$

$$\zeta(2-3T)x^2+(3-5T)x+\zeta(2T-1)=0$$
$$\sqrt{3}[2-3(0.189598041)]x^2+[3-5(0.189598041)]x+\sqrt{3}[2(0.189598041)-1]=0$$
$$2.478921295x^2+2.052009795x-1.075263928=0$$

Creating the *function* for the above equation *(Ref. Section 4.1)* expresses it as:

$$2.478921295x^2+2.052009795x-1.075263928=y$$

Taking the derivative of the above function with respect to y, and setting it equal to zero produces:

$$2(2.478921295)x_M+2.052009795=\frac{dy}{dx}$$
$$=0$$
$$2(2.478921295)x_M=-2.052009795$$
$$x_M=\frac{-2.052009795}{2(2.478921295)}$$
$$x_M=-0.413891679$$

$$2.478921293{x_M}^2+2.052009793x_M-1.075263926=y_M$$
$$2.478921293(-0.413891679)^2+2.052009793(-0.413891679)-1.075263926=y_M$$
$$-1.499918815=y_M$$

PROBLEM NUMBER 35 *(Ref. Section 14.1)*

GIVEN:

A *Quadratic Curve* whose

- Low or high point coordinates are:

  $x_M = -0.413891679$

  $y_M = -0.149991882$

- Root coordinates are:

  $x_1 = 0.363970234$

  $x_2 = -1.191753593$

DETERMINE:

The *Quadratic Function* which satisfies all of the above *given* conditions

SOLUTION:

Where,

- y_M denotes the *Quadratic Function's* minimum, or maximum, 'y' value
- x_M denotes the *Quadratic Function's* 'x' coordinate which corresponds to y_M
- x_1 denotes the *Quadratic Function's* first root
- x_2 denotes the *Quadratic Function's* second root

Considering a *Quadratic Function* of the following form:

$$y = ax^2 + bx + c$$

Differentiating with respect to x produces

$$\frac{dy}{dx} = 2ax + b$$

Since the low or high point on the curve occurs at point 'M' and exhibits zero slope, an expression in terms of x_M and the coefficient 'b' is obtained as follows:

$$\frac{dy}{dx} = 2ax_M + b = 0$$

Or,

$$b = -2ax_M$$

But,

$$\begin{aligned} y_M &= ax_M{}^2 + bx_M + c \\ &= ax_M{}^2 - 2ax_M{}^2 + c \\ &= c - ax_M{}^2 \end{aligned}$$

$$y_M + ax_M{}^2 = c$$

The last two expressions realized above then are substituted into the *Quadratic Function* to produce the following relationship:

$$
\begin{aligned}
y &= ax^2 + bx + c \\
&= ax^2 - 2ax_M x + (y_M + ax_M{}^2) \\
&= a(x^2 - 2x_M x + x_M{}^2) + y_M \\
&= a(x^2 - x_M)^2 + y_M
\end{aligned}
$$

At the location of the *first root* on this curve, the above function may be denoted as:

$$y_1 = a(x_1{}^2 - x_M)^2 + y_M$$

Where,

$$
\begin{aligned}
x_1 &= 0.363970234 \\
y_1 &= 0 \\
x_M &= -0.413891679 \\
y_M &= -0.149991882
\end{aligned}
$$

$$
\begin{aligned}
0 &= a(0.363970234 + 0.413891679)^2 - 0.149991882 \\
&= a(0.777861913)^2 - 0.149991882 \\
&= 0.605069156a - 0.149991882
\end{aligned}
$$

Then,

$$
\begin{aligned}
a &= \frac{-0.149991882}{0.605069156} \\
&= 0.24789213
\end{aligned}
$$

$$
\begin{aligned}
b &= -2a(-0.413891679) \\
&= 2(0.24789213)(0.413891679) \\
&= 0.205200979
\end{aligned}
$$

$$
\begin{aligned}
c &= y_M + a(-0.413891679)^2 \\
&= -0.149991882 + (0.24789213)(-0.413891679)^2 \\
&= -0.107526393
\end{aligned}
$$

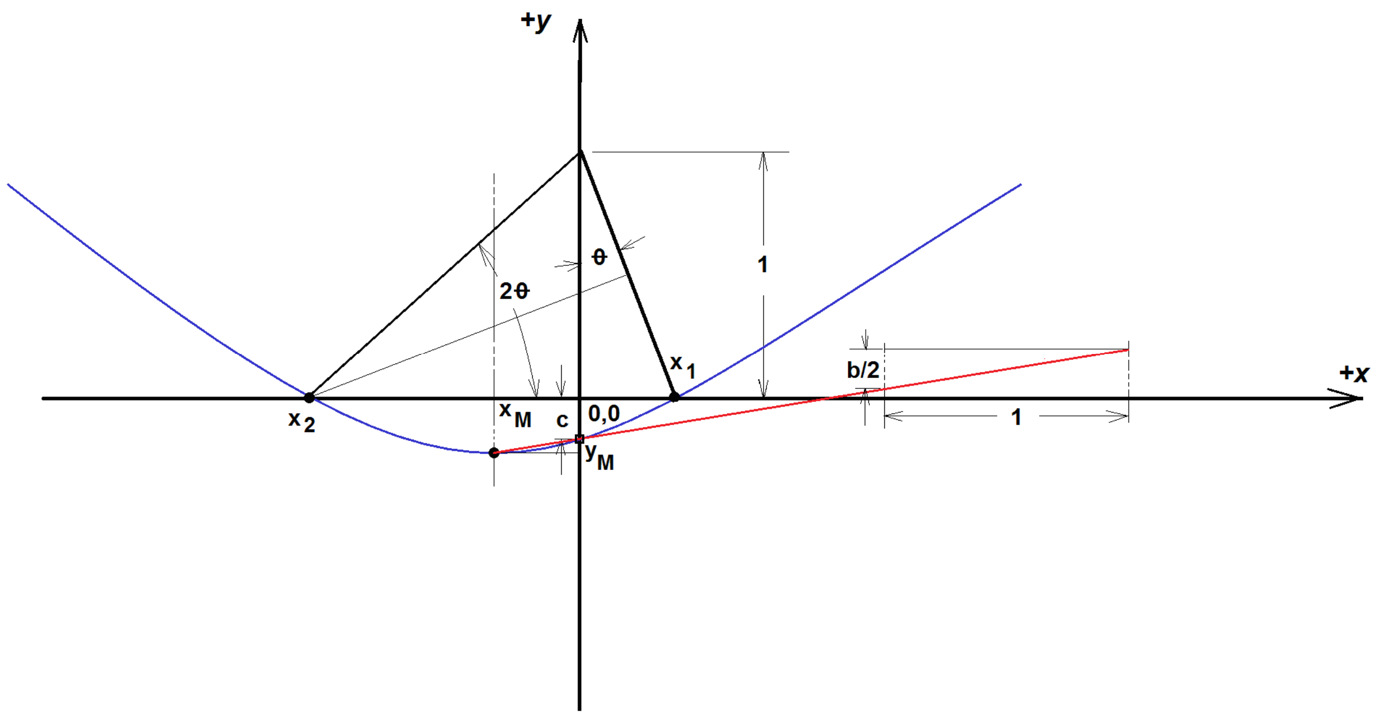
So, the *singular curve* which meets all of the above given conditions is:

$$0.24789213x^2 + 0.205200979x - 0.107526393 = y$$

The construction for this problem is shown *in Figure 68*. The height "c" designates the vertical offset with respect to the x-axis where the Quadratic Curve intersects the y-axis. This is confirmed in the following equation when the value "x" is set equal to zero:

$$
\begin{aligned}
y &= ax^2 + bx + c \\
&= a(0)^2 + b(0) + c \\
&= c
\end{aligned}
$$

Figure 68. Quadratic Curve Determined from its Low Point and Root Coordinates.



PROBLEM NUMBER 36 *(Ref. Section 14.1)*

GIVEN:

The following function *(Ref. last problem)*:

$$0.24789213x^2 + 0.205200979x - 0.107526393 = y$$

PLOT:

The *above given function* by applying ***RST Spreads*** instead of coefficients

DEPICT:

The *above given function* as the *cross-section* of the *critical throat area* of a *convoluted nozzle*

SOLUTION:

Plot:

Via *Quadratic Formula,* the locations of where the above given function crosses the x-axis are determined as follows,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-0.205200979 \pm \sqrt{(-0.205200979)^2 + 4(0.24789213)(0.107526393)}}{2(0.24789213)}$$

$$= \frac{-0.205200979 \pm \sqrt{0.148727228}}{2(0.24789213)}$$

$$x_{1;}x_2 = 0.363970234;\text{-}1.191753589$$

$$= \tan 20^o;\text{-}\frac{1}{\tan 40^o}$$

$$\zeta = \tan(3\theta) = \tan 60^o = \sqrt{3}$$

Comparison of the coefficients for each of the terms expressed in the *Unified Cubic Trigonometric Reduction Function (Ref. Equation 29)* shown below with those indicated in the *above given Quadratic Function* renders the following set of relationships:

[Ref. Equation 29]

$$\zeta(RST - 1) + [(R + S + T) - 3RST]\tan\theta + \zeta[(RS + RT + ST) - 3RST]\tan^2\theta = y$$

$$\zeta[(RS + RT + ST) - 3RST]\tan^2\theta + [(R + S + T) - 3RST]\tan\theta + \zeta(RST - 1) = y$$

$$ax^2 + b\,x + c = y$$

$$a = \zeta[(RS + RT + ST) - 3RST]$$

$$b = R + S + T - 3RST$$

$$c = \zeta(RST - 1)$$

Then,

$$RST - 1 = \frac{c}{\zeta}$$

$$RST = \frac{c+\zeta}{\zeta}$$

$$a = \zeta[(RS + RT + ST) - 3RST]$$

$$= \zeta[(RS + RT + ST) - 3(\frac{c+\zeta}{\zeta})]$$

But,

$$x_1 = \tan 20^o = R\tan\theta = R\tan 20^o$$

$$\tan 20^o = R\tan 20^o$$

$$1 = R$$

$$b = R + S + T - 3RST$$

$$= 1 + S + T - 3(\frac{c+\zeta}{\zeta})$$

$$= 0.2052009795$$

As such,

$$0.2052009795 = 1 + S + T - 3(\frac{c+\sqrt{3}}{\sqrt{3}})$$

$$-0.794799021 + 3(\frac{c+\sqrt{3}}{\sqrt{3}}) - T = S$$

$$-0.794799021 + 3(\frac{-0.1075263928+\sqrt{3}}{\sqrt{3}}) - T = S$$

$$-0.794799021 + 3(0.937919608) - T = S$$

$$-0.794799021 + 2.813758824 - T = S$$

$$2.018959804 - T = S$$

$$2.018959804 = S + T$$

But,

$$a = 0.2478921295 = \sqrt{3}[(RS + RT + ST) - 3(\frac{c+\sqrt{3}}{\sqrt{3}})]$$

$$= \sqrt{3}[1(S+T) + ST - 3(\frac{c+\sqrt{3}}{\sqrt{3}})]$$

$$= \sqrt{3}[(2.018959804) + (2.018959804 - T)T - 3(\frac{c+\sqrt{3}}{\sqrt{3}})]$$

$$= \sqrt{3}[(2.018959804) + (2.018959804 - T)T - 2.813758824]$$

$$= \sqrt{3}[(2.018959804 - T)T - 0.79479902]$$

$$0.2478921295 + \sqrt{3}(0.79479902) = \sqrt{3}T(2.018959804 - T)$$

$$0.2478921295 + 1.376632284 = \sqrt{3}T(2.018959804 - T)$$

$$\frac{1.624524413}{\sqrt{3}} = (2.018959804 - T)T$$

$$T^2 - 2.018959804T + 0.937919607 = 0$$

Therefore,

$$S;T = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2.018959804 \pm \sqrt{(-2.018959804)^2 - 4(1)(0.937919607)}}{2}$$

$$= \frac{2.018959804 \pm 0.569666797}{2}$$

$$= 1.294313301; 0.724646503$$

Check,

$$a = \sqrt{3}[R(S+T) + ST - 3(\frac{c+\sqrt{3}}{\sqrt{3}})]$$

$$= \sqrt{3}[1(\frac{2.018959804 + 0.569666797}{2} + \frac{2.018959804 - 0.569666797}{2}) + ST - 3(\frac{c+\sqrt{3}}{\sqrt{3}})]$$

$$= \sqrt{3}[2.018959804 + (\frac{2.018959804 + 0.569666797}{2})(\frac{2.018959804 - 0.569666797}{2}) - 3(\frac{c+\sqrt{3}}{\sqrt{3}})]$$

$$= \sqrt{3}[2.018959804 + \frac{4.07619869 - 0.324520259}{4} - 3(\frac{c+\sqrt{3}}{\sqrt{3}})]$$

$$= \sqrt{3}(2.018959804 + 0.937919607 - 2.813758824)$$

$$= \sqrt{3}(0.143120587)$$

$$= 0.2478921295]$$

The *plot* for the *above given function* is illustrated in *Figure 69*. It represents respective coordinates extracted from *Table 36* for the condition when $z = \tan\theta = \pm 1$.

Table 36 tabulates *coefficients* for the *above given function* in terms of the following ***RST Spreads***, the sums of which are equal to the ordinate values 'y':

$$a = \zeta[(RS + RT + ST) - 3RST]$$

$$b = R + S + T - 3RST$$

$$c = \zeta(RST - 1)$$

$$ax^2 + bx + c = y$$

Figure 69. Plot for the Function $y = 0.24789213x^2 + 0.205200979x - 0.107526393$.

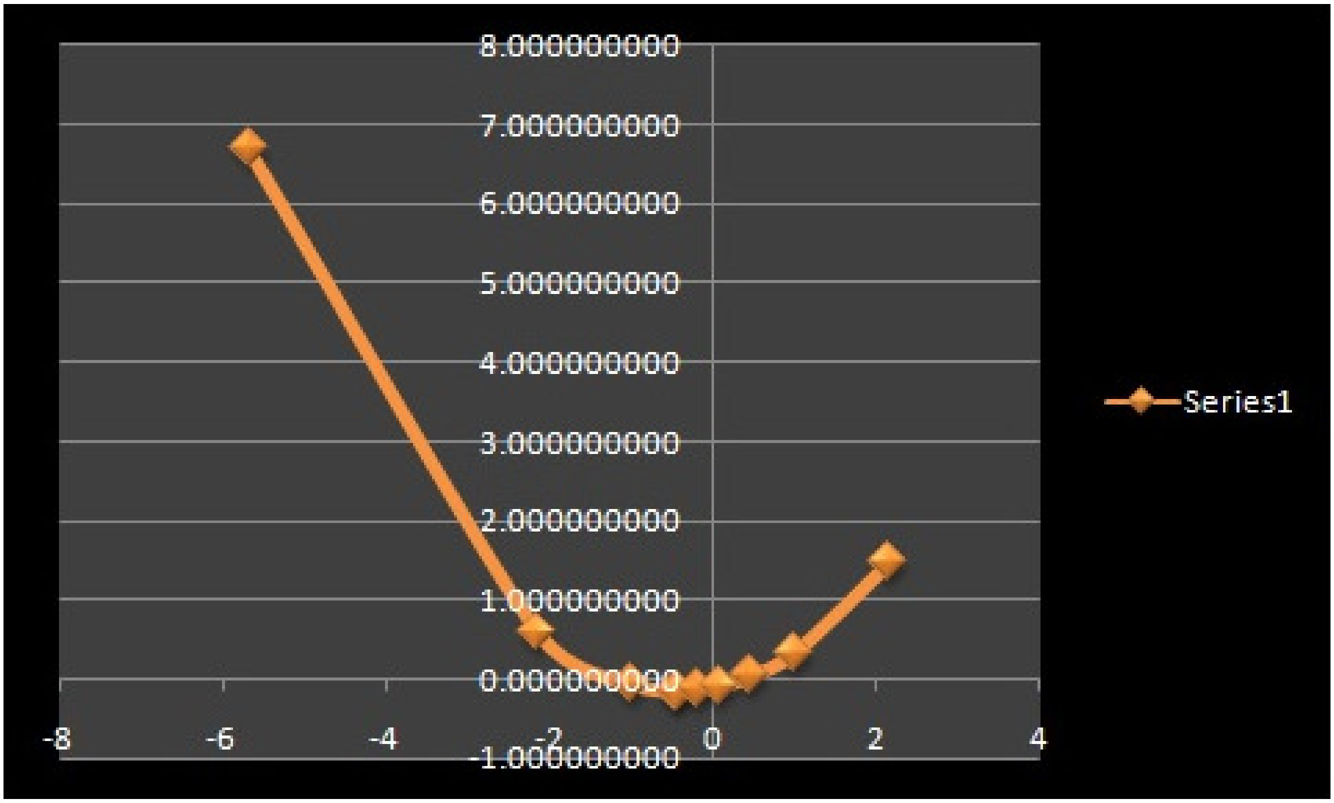


Table 36. Calculations to Determine Function Coefficients in terms of *RST Spreads*.

Θ Deg.	Z = tan (3θ) = $\sqrt{3}$	R	S	T	TAN θ z	*Coefficient a* ζ[(RS + RT + ST)-3RST] z^2	*Coefficient b* [(R+ S + T) – 3RST] z	*Coefficient c* ζ(RST – 1)	SUM Y
100	√3	1	1.294313301	0.724646503	-5.67128182	7.973063025	-1.163752587	-0.107526393	6.701784045
115	√3	1	1.294313301	0.724646503	-2.144506921	1.140033579	-0.440054922	-0.107526393	0.592452264
130	√3	1	1.294313301	0.724646503	-1.191753593	0.352075398	-0.244549005	-0.107526393	0.000000000
135	√3	1	1.294313301	0.724646503	-1	0.247892130	-0.205200980	-0.107526393	-0.064835243
155	√3	1	1.294313301	0.724646503	-0.466307658	0.053902367	-0.095686788	-0.107526393	-0.149310815
157.5157	√3	1	1.294313301	0.724646503	-0.413891679	0.042465489	-0.084930978	-0.107526393	-0.149991882
170	√3	1	1.294313301	0.724646503	-0.176326981	0.007707265	-0.036182469	-0.107526393	-0.136001598
5	√3	1	1.294313301	0.724646503	0.087488664	0.001897432	0.017952759	-0.107526393	-0.087676201
20	√3	1	1.294313301	0.724646503	0.363970234	0.032839344	0.074687048	-0.107526393	0.000000000
25	√3	1	1.294313301	0.724646503	0.466307658	0.053902367	0.095686788	-0.107526393	0.042062762
45	√3	1	1.294313301	0.724646503	1	0.247892130	0.205200980	-0.107526393	0.345566717
60	√3	1	1.294313301	0.724646503	1.732050808	0.743676390	0.355418523	-0.107526393	0.991568520
65	√3	1	1.294313301	0.724646503	2.144506921	1.140033579	0.440054922	-0.107526393	1.472562107

<u>Depiction</u>:

Table 37 shows many more increments than appear in *Table 36* in order to render a more accurate accounting from which to prepare the *cross-sectional plot* of the *convoluted nozzle's throat area*. Hence, line items for every five degrees are shown. The accompanying plot is given in *Figure 70*.

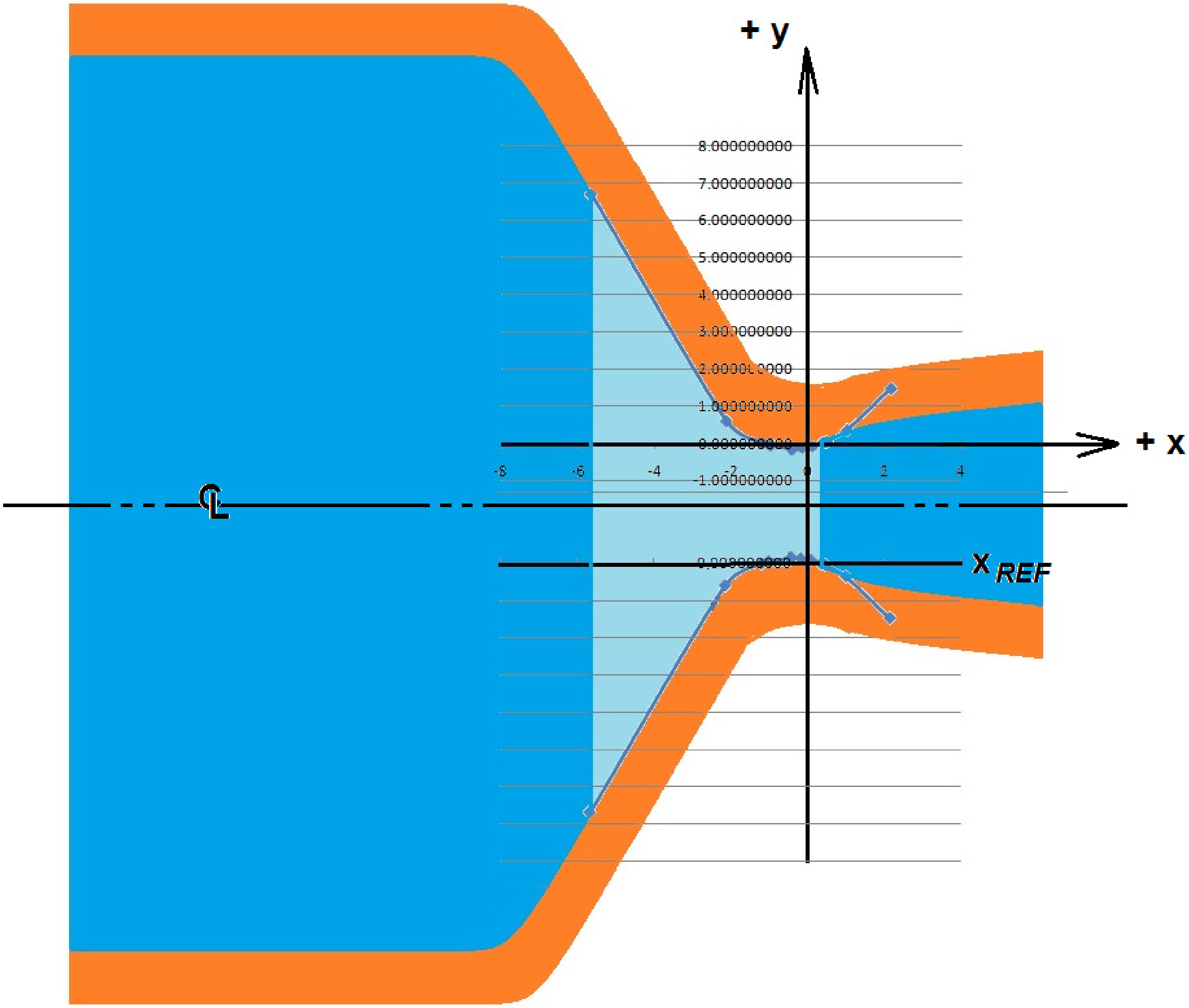
Theoretically speaking, a *three-dimensional program* can be generated from this depiction so that an actual working model of the nozzle can be produced.

Table 37. Detail Chart to Determine Function Coefficients in terms of *RST Spreads*.

θ Deg.	$\zeta = \tan(3\theta) = \sqrt{3}$	R	S	T	TAN θ z	$\zeta[(RS + RT + ST)-3RST]\ z^2$	$[(R+S+T) - 3RST]\ z$	$\zeta(RST - 1)$	SUM Y
95	√3	1	1.294313301	0.724646503	-11.4300523	32.386138929	-2.345457934	-0.107526393	29.933154602
100	√3	1	1.294313301	0.724646503	-5.67128182	7.973063025	-1.163752587	-0.107526393	6.701784045
105	√3	1	1.294313301	0.724646503	-3.732050808	3.452691966	-0.765820483	-0.107526393	2.579345090
110	√3	1	1.294313301	0.724646503	-2.747477419	1.871246507	-0.563785059	-0.107526393	1.199935055
115	√3	1	1.294313301	0.724646503	-2.144506921	1.140033579	-0.440054922	-0.107526393	0.592452264
120	√3	1	1.294313301	0.724646503	-1.732050808	0.743676390	-0.355418523	-0.107526393	0.280731474
125	√3	1	1.294313301	0.724646503	-1.428148007	0.505602457	-0.293057371	-0.107526393	0.105018693
130	√3	1	1.294313301	0.724646503	-1.191753593	0.352075398	-0.244549005	-0.107526393	0.000000000
135	√3	1	1.294313301	0.724646503	-1	0.247892130	-0.205200980	-0.107526393	-0.064835243
140	√3	1	1.294313301	0.724646503	-0.839099631	0.174537921	-0.172184067	-0.107526393	-0.105172538
145	√3	1	1.294313301	0.724646503	-0.700207538	0.121539180	-0.143683273	-0.107526393	-0.129670486
150	√3	1	1.294313301	0.724646503	-0.577350269	0.082630710	-0.118472841	-0.107526393	-0.143368524
155	√3	1	1.294313301	0.724646503	-0.466307658	0.053902367	-0.095686788	-0.107526393	-0.149310815
157.5157	√3	1	1.294313301	0.724646503	-0.413891679	0.042465489	-0.084930978	-0.107526393	-0.149991882
160	√3	1	1.294313301	0.724646503	-0.363970234	0.032839344	-0.074687049	-0.107526393	-0.149374098
165	√3	1	1.294313301	0.724646503	-0.267949192	0.017797854	-0.054983437	-0.107526393	-0.144711976
170	√3	1	1.294313301	0.724646503	-0.176326981	0.007707265	-0.036182469	-0.107526393	-0.136001598
175	√3	1	1.294313301	0.724646503	-0.087488664	0.001897432	-0.017952759	-0.107526393	-0.123581720
180	√3	1	1.294313301	0.724646503	0	0.000000000	0.000000000	-0.107526393	-0.107526393
0	√3	1	1.294313301	0.724646503	0	0.000000000	0.000000000	-0.107526393	-0.107526393
5	√3	1	1.294313301	0.724646503	0.087488664	0.001897432	0.017952759	-0.107526393	-0.087676201
10	√3	1	1.294313301	0.724646503	0.176326981	0.007707265	0.036182469	-0.107526393	-0.063636659
15	√3	1	1.294313301	0.724646503	0.267949192	0.017797854	0.054983437	-0.107526393	-0.034745102

θ Deg.	ζ = tan (3θ) = $\sqrt{3}$	R	S	T	TAN θ z	ζ[(RS + RT + ST)-3RST] z^2	[(R+ S + T) – 3RST] z	ζ(RST - 1)	SUM Y
20	√3	1	1.294313301	0.724646503	0.363970234	0.032839344	0.074687048	-0.107526393	0.000000000
25	√3	1	1.294313301	0.724646503	0.466307658	0.053902367	0.095686788	-0.107526393	0.042062762
30	√3	1	1.294313301	0.724646503	0.577350269	0.082630710	0.118472841	-0.107526393	0.093577158
35	√3	1	1.294313301	0.724646503	0.700207538	0.121539180	0.143683273	-0.107526393	0.157696060
40	√3	1	1.294313301	0.724646503	0.839099631	0.174537921	0.172184067	-0.107526393	0.239195595
45	√3	1	1.294313301	0.724646503	1	0.247892130	0.205200980	-0.107526393	0.345566717
50	√3	1	1.294313301	0.724646503	1.191753593	0.352075398	0.244549005	-0.107526393	0.489098010
55	√3	1	1.294313301	0.724646503	1.428148007	0.505602457	0.293057371	-0.107526393	0.691133434
60	√3	1	1.294313301	0.724646503	1.732050808	0.743676390	0.355418523	-0.107526393	0.991568520
65	√3	1	1.294313301	0.724646503	2.144506921	1.140033579	0.440054922	-0.107526393	1.472562107
70	√3	1	1.294313301	0.724646503	2.747477419	1.871246507	0.563785059	-0.107526393	2.327505173
75	√3	1	1.294313301	0.724646503	3.732050808	3.452691966	0.765820483	-0.107526393	4.110986056
80	√3	1	1.294313301	0.724646503	5.67128182	7.973063025	1.163752587	-0.107526393	9.029289219
85	√3	1	1.294313301	0.724646503	11.4300523	32.386138929	2.345457934	-0.107526393	34.624070470

Figure 70. Detail Plot of the Cross-section of a Convoluted Nozzle's Throat Area.



CIRCULAR CROSS-SECTION

PROBLEM NUMBER 37 *(Ref. Section 14.1)*

GIVEN:

An *unknown Quadratic Curve* has:

- A root designated as x_1 located at the origin
- A *low point* which resides the same exact distance away from coordinate (0,1) as its *other root*, x_2

DETERMINE:

The coefficients for the *unknown Quadratic Curve*

SOLUTION:

Firstly, since one of the *roots* of the *unknown Quadratic Curve* resides upon the origin, it must assume the form:

$$ax^2 + bx = y$$

This is because:

$$ax^2 + bx + c = y$$
$$a(0)^2 + b(0) + c = 0$$
$$c = 0$$

Secondly, the *low point* and the *other root* of the *unknown Quadratic Curve* lie upon the circumference of a circle of designated radius 'R' whose center is located at coordinate (0,1). Hence, the circle must assume the following form:

$$x^2 + (y-1)^2 = R^2$$

Letting '2θ' represent the angle which the radius to this *other root* makes with the y-axis, then *(Ref. Figure 71)*:

$$x_2 = \tan(2\theta)$$

$$R = \frac{1}{\cos(2\theta)}$$

Furthermore, such low point maintains an x-coordinate whose value is one-half that of the x_2 root as follows:

$$x_M = \frac{\tan(2\theta)}{2}$$

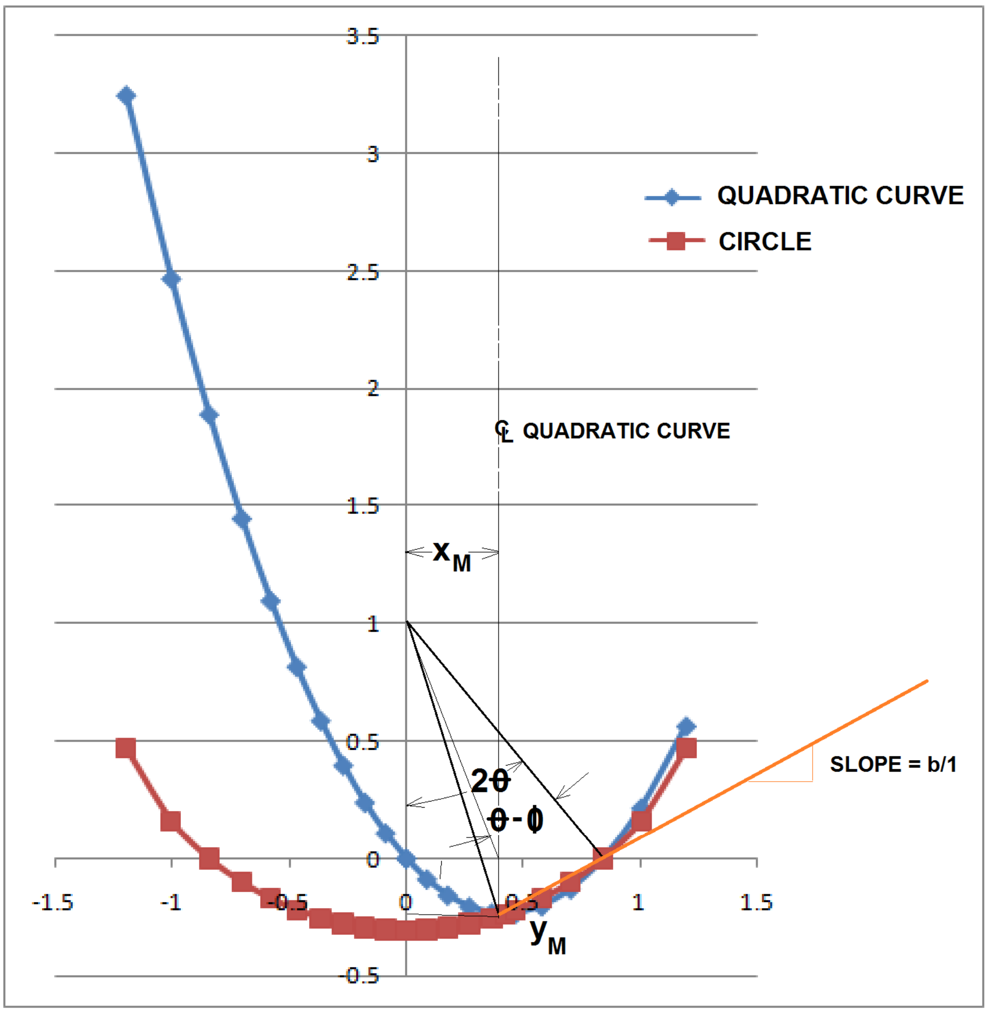
This is because the $ax^2 = y$ parabola exhibits a low point which lies on a vertical centerline which exists midway between any other point on the curve and an associated third point along the curve which possesses the same y-value as the second point.

With respect to *Figure 71*, it is easily recognized that this problem has multiple solutions. As the value 'R' increases, so does the magnitude of the angle '2θ'. This in turn requires that the *unknown Quadratic Curve* becomes wider in order to accommodate a larger value of $x_2 = \tan(2\theta)$.

As such, this problem is solved for the particular case when θ equals 20° <u>only</u>, which computes to:

$$x_M = \frac{\tan(2\theta)}{2}$$
$$= \frac{0.839099631}{2}$$
$$= 0.419549815$$

Figure 71. Circle and Quadratic Curve Intersection Points.



$a x^2 + b x + c = y$ Where, c = 0 $1.34159174 x^2 - 1.125729134 x = y$

QUADRATIC CURVE

$R^2 = x^2 + (y - 1)^2$

CIRCLE

$\tan \phi = \tan^3 \theta$

Then, y_M may be calculated as follows:

$$x_M{}^2 + (y_M - 1)^2 = R^2$$

$$(y_M - 1)^2 = R^2 - x_M{}^2$$

$$y_M = 1 \pm \sqrt{R^2 - x_M{}^2}$$

$$= 1 \pm \sqrt{[\frac{1}{\cos(2\theta)}]^2 - [\frac{\tan(2\theta)}{2}]^2}$$

$$= 1 \pm \sqrt{[\frac{1}{\cos(2\theta)}]^2 - [\frac{\sin(2\theta)}{2\cos(2\theta)}]^2}$$

$$= 1 \pm \frac{1}{2\cos(2\theta)}\sqrt{4 - \sin^2(2\theta)}$$

$$= 1 \pm 1.236149725$$

$$= 2.236149725; -0.236149725$$

Since,

$c = y_M + ax_M{}^2$ *(Ref. Problem Number 35)*

$$0 = -0.236149725 + a(0.419549815)^2$$

$$-a = \frac{-0.236149725}{(0.419549815)^2}$$

$$a = 1.341591738$$

For,

$$ax^2 + bx = 0$$

$$ax + b = 0$$

$$b = -ax$$

$$= -a\tan(2\theta)$$

$$= -1.341591738(0.839099631)$$

$$= -1.125729134$$

Hence, the *Quadratic Curve* assumes the following form:

$$1.34159174x^2 - 1.125729134x = y$$

Where,

$$R = \frac{1}{\cos(2\theta)} = 1.305407289$$

As portrayed in *Figure 71*, a *straight line* is drawn from coordinate (0,1) to a point which lies midway between the origin and the root x_2. This line is said to form an angle of $\theta+\varphi$ with the y-axis such that:

$$\tan\phi = \tan^3\theta$$

Where,

$$2\theta = (\theta + \phi) + (\theta - \phi)$$

Lastly, *Table 38* represents the associated graph from which *Figure 71* was plotted.

Table 38. Circle and Associated Quadratic Curve Plot.

2θ Deg.	$x = \tan(2\theta)$	a	b	c = 0	$y_{QuadEq} = ax^2 + bx$	$y_{Circle} = 1 \pm \sqrt{R^2 - x_M{}^2}$	
130	-1.191753593	1.34159174	-1.125729134	0	3.247023129	0.467260321	
135	-1	1.34159174	-1.125729134	0	2.467320874	0.160900369	
140	-0.839099631	1.34159174	-1.125729134	0	1.889197802	0	
145	-0.700207538	1.34159174	-1.125729134	0	1.44601384	-0.101724827	
150	-0.577350269	1.34159174	-1.125729134	0	1.097137265	-0.170792406	
155	-0.466307658	1.34159174	-1.125729134	0	0.816655624	-0.219280673	
160	-0.363970234	1.34159174	-1.125729134	0	0.587458365	-0.253640243	
165	-0.267949192	1.34159174	-1.125729134	0	0.397960166	-0.277611608	
170	-0.176326981	1.34159174	-1.125729134	0	0.240208122	-0.293443848	
175	-0.087488664	1.34159174	-1.125729134	0	0.108757438	-0.302472236	
180	0	1.34159174	-1.125729134	0	0	-0.305407289	
0	0	1.34159174	-1.125729134	0	0	-0.305407289	
5	0.087488664	1.34159174	-1.125729134	0	-0.088219637	-0.302472236	
10	0.176326981	1.34159174	-1.125729134	0	-0.156784717	-0.293443848	
15	0.267949192	1.34159174	-1.125729134	0	-0.205316259	-0.277611608	
20	0.363970234	1.34159174	-1.125729134	0	-0.232005428	-0.253640243	
22.7604763	0.419549815	1.34159174	-1.125729134	0	-0.236149725	-0.236149725	Y_M
25	0.466307658	1.34159174	-1.125729134	0	-0.233216609	-0.219280673	
30	0.577350269	1.34159174	-1.125729134	0	-0.202742772	-0.170792406	
35	0.700207538	1.34159174	-1.125729134	0	-0.130474211	-0.101724827	
40	0.839099631	1.34159174	-1.125729134	0	0	0	
45	1	1.34159174	-1.125729134	0	0.215862606	0.160900369	
50	1.191753593	1.34159174	-1.125729134	0	0.56383965	0.467260321	

PROBLEM NUMBER 38 *(Ref. Section 14.1)*

GIVEN:

Two sister stars almost identically sized at about one-fifth the diameter of the sun are located 73.861885613 million miles apart from one another. Sparked by the fact that they have aroused enormous interest here on earth, a drone flight has been slated to perform reconnaissance operations. The drone is scheduled to pass *exactly midway* between the sister stars, and also, at a different time, to photograph a total eclipse of the sun by the earth.

The coordinate system origin is to be located on the rightmost sister star where the x-axis is to form a 45° angle above a straight line drawn between the two stars.

For viewing the eclipse, the drone is to be at the following specified location with respect to the newly established origin.

$x_M = \text{-}4{,}156{,}459.589$ miles

$y_M = \text{-}103{,}256{,}336.854$ miles

Or,

$x_M = \text{-}4.156459589$ million miles

$x_M = \text{-}103.256336854$ million miles

The drone mission is to employ a *parabolic trajectory* whose *low point* is where the actual photographing of the eclipse of the sun is to take place.

DETERMINE:

a) The exact straight line distance between the drone and the origin when it reaches the parabola's low point;

b) The distance away from the origin when the drone is perfectly aligned between the two stars;

c) The coordinates of the drone when it is perfectly aligned between the two stars;

d) The *equation of the parabola*;

e) The exact straight line distance between the drone and the origin when it crosses its x-axis;

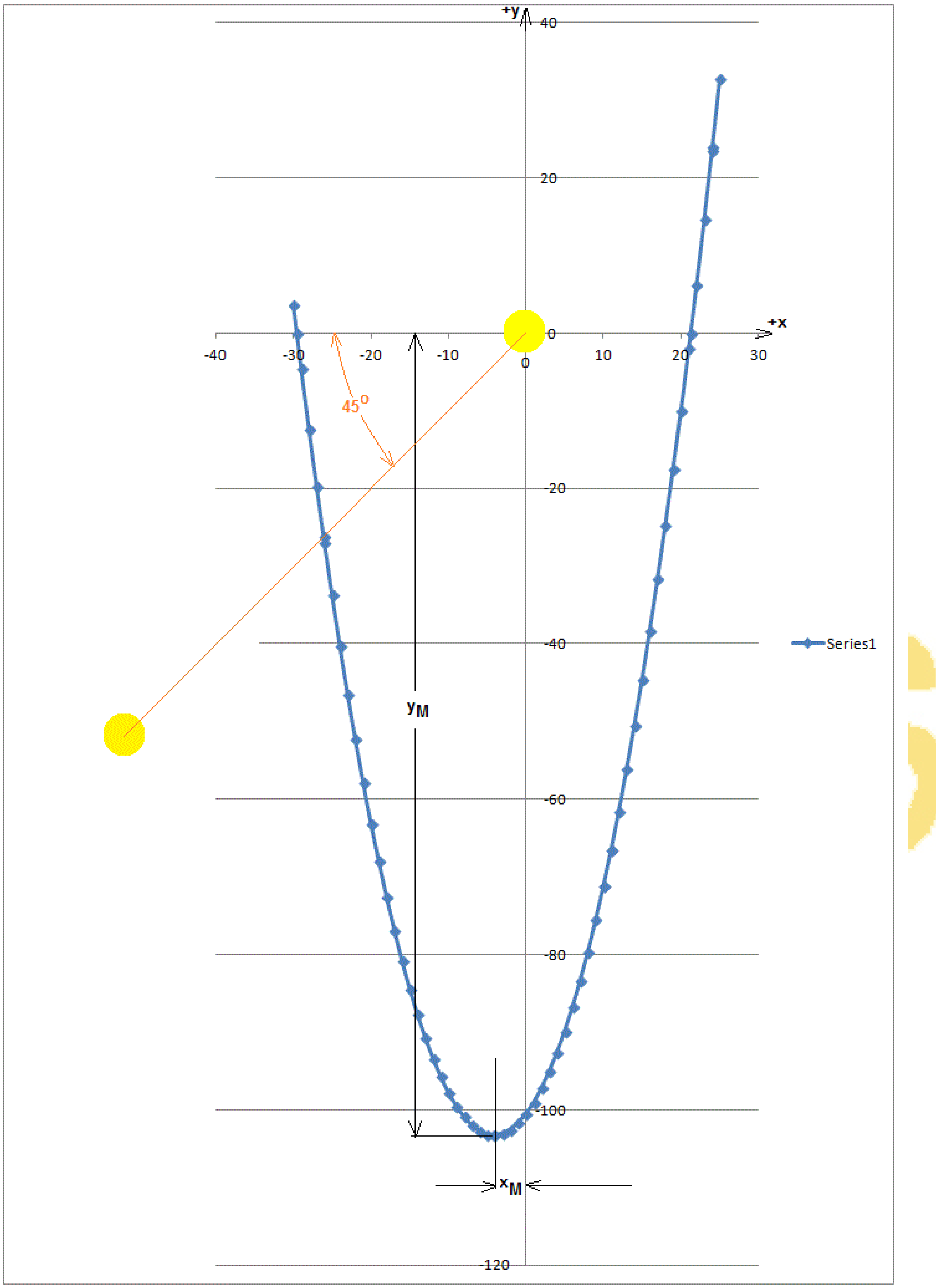
f) The coordinates of other locations on the parabola during the mission; and

g) Demonstrate the journey of the drone.

SOLUTION:

Figure 72, presented below, illustrates the *given problem*.

Figure 72. Navigation Problem.



SOLUTION:

a) Via Pythagorean Theorem:

$$d = \sqrt{x_M{}^2 + y_M{}^2}$$
$$= \sqrt{(-4.156459589)^2 + (-103.256336854)^2}$$
$$= 103.3399596 \text{ million miles}$$

b) $\dfrac{73.86188561}{2} = 36.9309428$ million miles

c) $\sin 225^o = \cos 225^o = -\dfrac{\sqrt{2}}{2} = \dfrac{x}{36.9309428} = \dfrac{y}{36.9309428}$

$$-\frac{\sqrt{2}}{2}(36.9309428) = x = y$$
$$-26.11412009 =$$

The respective coordinates are $(-26.11412009;-26.11412009)$ million miles.

d) The equation for a parabola whose origin is located directly at its low point is as follows:

$$y'' = ax''^2$$

Its *associated equation* about an origin located directly above its low point is as follows:

$$y' = ax'^2 + y_M$$

Where $x' = x''$

When this new origin again is moved to the right an absolute distance x_M, the resulting coordinates of the same curve are as follows:

$$x = x' + x_M$$
$$y = y'$$

Then, by substituting this information into the above equation, the following relationship is obtained:

$$y' = ax'^2 + y_M$$
$$y = a(x - x_M)^2 + y_M$$
$$= a(x^2 - 2x_M x + x_M{}^2) + y_M$$
$$= ax^2 - (2ax_M)x + (ax_M{}^2 + y_M)$$
$$y = ax^2 - (2ax_M)x + (ax_M{}^2 + y_M)$$
$$-26.11412009 = a(-26.11412009)^2 - (2a)(-4.156459589)(-26.11412009) + a(-4.156459589)^2 - 103.256336854$$

$$77.14221681 = 482.1388547a$$

$$\frac{77.14221681}{482.1388547} = a$$

$$0.16 =$$

$$b = -2ax_M$$
$$= -(2(0.16)(-4.156459589)$$
$$= +1.330067068$$

$$c = ax_M{}^2 + y_M$$
$$= (0.16)(-4.156459589)^2 \text{ -}103.256336854$$
$$= \text{-}100.4921518$$

$$y = ax^2 + bx + c$$
$$= 0.16x^2 + 1.330067068x - 100.4921518$$

e) $x = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$

$$= \frac{1}{2(0.16)}[-1.330067068 \pm \sqrt{(1.330067068)^2 - 4(0.16)(-100.4921518)}]$$

$x_1; x_2 = (21.24732173; \text{-}29.56024091)$ million miles

f) *Table 39* gives coordinates for other locations resident upon the *Figure 72 parabola.*

Table 39. Coordinates of Other Locations on the Parabola During the Mission.

x	$y = 0.16x^2 + 1.330067068x - 100.4921518$	x	$y = 0.16x^2 + 1.330067068x - 100.4921518$
25	32.75952487	-4	-103.2524201
24.05120091	24.05120091	-4.156459589	-103.2563369
24	23.5894578	-5	-103.1424872
23	14.73939073	-6	-102.7125543
22	6.209323663	-7	-101.9626213
21.24732173	0	-8	-100.8926884
21	-2.000743406	-9	-99.50275546
20	-9.890810474	-10	-97.79282253
19	-17.46087754	-11	-95.7628896
18	-24.71094461	-12	-93.41295667
17	-31.64101168	-13	-90.74302373
16	-38.25107875	-14	-87.7530908
15	-44.54114582	-15	-84.44315787
14	-50.51121288	-16	-80.81322494
13	-56.16127995	-17	-76.86329201
12	-61.49134702	-18	-72.59335908
11	-66.50141409	-19	-68.00342614
10	-71.19148116	-20	-63.09349321
9	-75.56154823	-21	-57.86356028
8	-79.6116153	-22	-52.31362735
7	-83.34168236	-23	-46.44369442
6	-86.75174943	-24	-40.25376149
5	-89.8418165	-25	-33.74382856
4	-92.61188357	-26	-26.91389562
3	-95.06195064	-26.11412009	-26.11412009
2	-97.19201771	-27	-19.76396269
1	-99.00208478	-28	-12.29402976
0	-100.4921518	-29	-4.50409683
-1	-101.6622189	-29.56024091	0
-2	-102.512286	-30	3.605836102
-3	-103.042353		

g) Animation may be viewed by *clicking* on the *Parabolic Trajectory.html* file available on the *hard disk* which this treatise file is located on. Clicking should initiate another enclosed animation file to play directly on your *default browser.*

PROBLEM NUMBER 39 *(Ref. Section 14.1.2)*

GIVEN:

The following *Parabolic Function*:

$$3x^2 - 200 = y$$

LOCATE:

A *series of Parabolic Curves* each of which bear the *exact curve shape* and *same low point value* as the *given curve*, but whose respective *generated root sets* along the x-axis exhibit the following *three root structures*:

$$x_1 = \tan\theta$$
$$x_2 = -1/\tan(2\theta)$$

$$x_1 = \tan\theta$$
$$x_2 = -1/\tan\theta$$

$$x_1 = \tan\theta$$
$$x_2 = -\tan(2\theta)$$

SOLUTION:

The *unknown curves* are to be represented by the following *Parabolic* form:

$$3x^2 + bx + c = y$$

Now, where:

$$3x^2 - 200 = y$$

At $y = 0$

$$3x^2 - 200 = 0$$

$$\bar{x} = \pm\sqrt{\frac{200}{3}}$$
$$= \pm 8.164965809$$

For the *first set of conditions*,

- $x_1 = \tan\theta$
- $x_2 = -1/\tan(2\theta)$

Applying:

$x_1 - x_2 = 2\bar{x}$ *(Ref. List of Properties Section 14.1.2)*

$$\tan\theta + (\frac{1 - \tan^2\theta}{2\tan\theta}) = 2\bar{x}$$

$$\frac{2\tan^2\theta + 1 - \tan^2\theta}{2\tan\theta} = 2\bar{x}$$

$$\frac{\tan^2\theta + 1}{2\tan\theta} = 2\bar{x}$$

Multiplying both sides of the equation by a factor of $2\tan\theta$:

$$\tan^2\theta + 1 = 2\bar{x}(2\tan\theta)$$

$$\tan^2\theta - (4\bar{x})\tan\theta = -1$$

Completing the Square produces:

$$\tan^2\theta - (4\bar{x})\tan\theta + (2\bar{x})^2 = 4\bar{x}^2 - 1$$

$$(\tan\theta - 2\bar{x})^2 = 4\bar{x}^2 - 1$$

$$(\tan\theta - 2\sqrt{\frac{200}{3}})^2 = 4(\frac{200}{3}) - 1$$

$$\tan\theta = 2\sqrt{\frac{200}{3}} \pm \sqrt{\frac{800-3}{3}}$$

$$= \frac{\sqrt{3}}{3}(\sqrt{800} \pm \sqrt{797})$$

$$x_1 = \tan\theta = 32.62921586; +0.03064738$$

$$\theta = 88.24458389^o; 1.755416107^o$$

$$2\theta = 176.4891678^o; 3.510832213^o$$

$$\tan(2\theta) = -0.061352387; 0.061352387$$

$$x_2 = -1/\tan(2\theta)$$
$$= +1/0.061352387; -1/0.061352387$$
$$= +16.29928423; -16.29928423$$

Check,

$$2\bar{x} = x_1 - x_2$$
$$= (32.62921586; +0.03064738) - (16.29928423; -16.29928423)$$
$$= 16.32993162$$
$$\bar{x} = 8.164965809$$

Now, since:

$$\bar{x} = x_1 - x_M \qquad \text{[Ref. Figure 13]}$$

$$\begin{aligned} x_M &= x_1 - \bar{x} \\ &= (32.62921586;+0.03064738) - 8.164965809 \\ &= 24.46425005;-8.134318429 \end{aligned}$$

Verification,

$$\bar{x} = x_M - x_2 \qquad \text{[Ref. Figure 13]}$$

$$\begin{aligned} x_M &= x_2 + \bar{x} \\ &= (16.29928423;-16.29928423) + 8.164965809 \\ &= 24.46425005;-8.134318429 \end{aligned}$$

Now where,

$$\begin{aligned} b &= -2ax_M \\ &= -2(3)(24.46425005;-8.134318429) \\ &= -146.7855003;+48.80591057 \end{aligned}$$

$$a(x_M)^2 + bx_M + c = y_M$$

$$a(x_M)^2 - 2a(x_M)^2 + c = -200$$

$$\begin{aligned} c &= a(x_M)^2 - 200 \\ &= 3(24.46425005;-8.134318429)^2 - 200 \\ &= 3(+598.4995305;+66.1671363) - 200 \\ &= (1795.498592;+198.5014089) - 200 \\ &= 1595.498592;-1.498591087 \end{aligned}$$

Then,

$$3x^2 - 146.7855003x + 1595.498592 = y$$

$$3x^2 + 48.80591057x - 1.498591087 = y$$

Check,

$$\frac{b}{a} = -(x_1 + x_2) \qquad \text{(Ref. small table Section 14.1.2)}$$

$$\frac{-146.7855003}{3} = -[(32.62921586;+0.03064738) + (16.29928423;-16.29928423)]$$

$$-48.9285001 = -48.9285001;+16.26863685$$

$$\frac{48.80591057}{3} = -[(32.62921586;+0.03064738) + (16.29928423;-16.29928423)]$$

$$+16.26863685 = -48.9285001;+16.26863685$$

And,

$$\frac{b}{a}=\frac{1-3\tan^2\theta}{2\tan\theta}$$ *(Ref. small table Section 14.1.2)*

$$=\frac{1-3(32.62921586)^2}{2(32.62921586)};\frac{1-3(0.03064738)^2}{2(0.03064738)}$$
$$=\frac{1-3(1064.665728)}{2(32.62921586)};\frac{1-3(0.000939261)}{2(0.03064738)}$$
$$=\frac{1-3193.997183}{65.25843172};\frac{1-0.002817785}{0.06129476}$$
$$=\frac{-3192.997183}{65.25843172};\frac{0.997182214}{0.061468621}$$
$$=-48.9285001;16.26863685$$

For the *second set of conditions*,

- $x_1=\tan\theta$
- $x_2=-1/\tan\theta=-1/x_1$

Applying:

$$x_1 - x_2 = 2\bar{x}$$ *(Ref. List of Properties Section 14.1.2)*

$$x_1+\frac{1}{x_1}=2\bar{x}$$
$$\frac{x_1^{\ 2}+1}{x_1}=2\bar{x}$$

Multiplying both sides of the equation by a factor of x_1:

$$x_1^{\ 2}+1=(2\bar{x})x_1$$
$$x_1^{\ 2}-(2\bar{x})x_1=-1$$

Completing the Square produces:

$$x_1^{\ 2}-(2\bar{x})x_1+(\bar{x})^2=(\bar{x})^2-1$$
$$(x_1-\bar{x})^2=(\bar{x})^2-1$$
$$(x_1-\sqrt{\frac{200}{3}})^2=(\frac{200}{3})-1)(\frac{3}{3})$$
$$x_1=\sqrt{\frac{200}{3}}\pm\sqrt{\frac{200-3}{3}}$$
$$=\frac{\sqrt{3}}{3}(\sqrt{200}\pm\sqrt{197})$$
$$x_1=+16.268463;+0.061468621$$

$$x_2=-1/x_1$$
$$=-1/16.268463;-1/0.061468621$$
$$=-0.061468621;-16.268463$$

Check,

$$2\bar{x} = x_1 - x_2$$
$$= (16.268463;+0.061468621) - (-0.061468621;-16.268463)$$
$$= 16.32993162$$
$$\bar{x} = 8.164965809$$

$\bar{x} = x_1 - x_M$ *[Ref. Figure 13]*

$$x_M = x_1 - \bar{x}$$
$$= (16.268463;+0.061468621) - 8.164965809$$
$$= 8.103497191;-8.103497191$$

Verification,

$\bar{x} = x_M - x_2$ *[Ref. Figure 13]*

$$x_M = x_2 + \bar{x}$$
$$= (-0.061468621;-16.268463) + 8.164965809$$
$$= 8.103497191;-8.103497191$$

Where,

$$b = -2ax_M$$
$$= -2(3)(8.103497191;-8.103497191)$$
$$= 48.62098315;-48.62098315$$

$$c = a(x_M)^2 - 200$$
$$= 3(\pm 8.103497191)^2 - 200$$
$$= 3(65.666667) - 200$$
$$= 197 - 200$$
$$= -3$$

Then,

$$3x^2 \pm 48.62098315x - 3 = y$$

Check,

$\frac{b}{a} = -(x_1 + x_2)$ *(Ref. small table Section 14.1.2)*

$$\frac{\pm 48.62098315}{3} = -[(16.268463;+0.061468621) + (-0.061468621;-16.268463)]$$
$$\pm 16.20699438 = \pm 16.20699438$$

Or,

$\frac{b}{a} = \frac{1 - \tan^2\theta}{\tan\theta}$ *(Ref. small table Section 14.1.2)*

$$= \frac{1 - x_1{}^2}{x_1}$$

$$= \frac{1-(16.268463)^2}{16.268463}; \frac{1-(0.061468621)^2}{0.061468621}$$

$$= \frac{1-264.6628884}{16.268463}; \frac{1-0.003778391}{0.061468621}$$

$$= \frac{-263.6628884}{16.268463}; \frac{0.996221609}{0.061468621}$$

$$= \pm 16.20699438$$

For the *third set of conditions*,

- $x_1 = \tan\theta$
- $x_2 = -\tan(2\theta)$

Applying:

$$x_1 - x_2 = 2\bar{x}$$ *(Ref. List of Properties Section 14.1.2)*

$$\tan\theta + \frac{2\tan\theta}{1 - \tan^2\theta} = 2\bar{x}$$

$$\frac{\tan\theta(1 - \tan^2\theta) + 2\tan\theta}{1 - \tan^2\theta} = 2\bar{x}$$

$$\frac{3\tan\theta - \tan^3\theta}{1 - \tan^2\theta} = 2\bar{x}$$

$$\frac{3x_1 - x_1{}^3}{1 - x_1{}^2} = 2\bar{x}$$

Multiplying both sides of the equation by a factor of $1 - x_1{}^2$:

$$3x_1 - x_1{}^3 = 2\bar{x}(1 - x_1{}^2)$$

Then,

$$x_1{}^3 - 2\bar{x}x_1{}^2 - 3x_1 + 2\bar{x} = 0$$

Substituting $x_1 = u$ into the *above given equation* renders:

$$u^3 \quad 2\bar{x}u^2 \quad 3u + 2\bar{x} = 0$$

Equation 42 is selected from the excerpt since it matches the above format,

$$u^3 + (3V)u^2 + 3(V^2 - f^2)u + V^3 - 3f^2V - 2\psi f^3 = 0$$ *[Ref. Equation 42]*

Comparing respective coefficients with those of the *given 3θ Cubic Equation* returns the following results,

$$V = \frac{-2\bar{x}}{3} = -\sqrt{\frac{4(200)}{9(3)}} = -\sqrt{\frac{800}{27}} = -5.44331054$$
$$f = \sqrt{V^2 + 1} = 5.53440418$$
$$\psi = \frac{V^3 - 3f^2V - 2\bar{x}}{2f^3} = 0.951429726 = \cos(6\omega)$$

Since $-1 \le \cos(6\omega) \le +1$ the *three roots are all real*. Hence,

$$6\omega = +17.93067295^o; -17.93067295^o; +(17.93067295^o + 360^o)$$
$$= +17.93067295^o; +342.0693271^o; +377.930673^o$$

$$2\omega = \frac{+17.93067295^o}{3}; \frac{+342.0693271^o}{3}; \frac{+377.930673^o}{3}$$
$$= +5.976890982^o; +114.023109^o; +125.976891^o$$

$$\cos(2\omega) = +0.994563973; -0.407105068 - 0.587458904$$

$$2f\cos(2\omega) = (2f)(+0.994563973; -0.407105068 - 0.587458904)$$
$$= (11.06880836)(+0.994563973; -0.407105068 - 0.587458904)$$
$$= +11.00863802; -4.506167989; -6.502470038$$
$$= \ell$$

Now since,

$$u = \ell - V$$
$$= (+11.00863802; -4.506167989; -6.502470038) - (-5.44331054)$$
$$x_1 = +16.45194856, +0.93714255; -1.059159498$$

Check,

$$x_R{}^3 - 2\bar{x}x_R{}^2 - 3x_R + 2\bar{x} = (16.45194856)^3 - 2\sqrt{\frac{200}{3}}(16.45194856)^2 - 3(16.45194856) + 2\sqrt{\frac{200}{3}}$$
$$= 4452.993168 - 4419.967256 - 49.35584568 + 16.32993162$$
$$= 4469.32310 - 4469.32310$$
$$= 0$$

$$x_S{}^3 - 2\bar{x}x_S{}^2 - 3x_S + 2\bar{x} = (0.93714255)^3 - 2\sqrt{\frac{200}{3}}(0.93714255)^2 - 3(0.93714255) + 2\sqrt{\frac{200}{3}}$$
$$= 0.823032475 - 14.34153644 - 2.811427652 + 16.32993162$$
$$= 17.15296409 - 17.15296409$$
$$= 0$$

$$x_T{}^3 - 2\bar{x}x_T{}^2 - 3x_T + 2\bar{x} = (-1.059159498)^3 - 2\sqrt{\frac{200}{3}}(-1.059159498)^2 - 3(-1.059159498) + 2\sqrt{\frac{200}{3}}$$
$$= -1.188185082 - 18.31922499 + 3.177478494 + 16.32993162$$
$$= -19.50741011 + -19.50741011$$
$$= 0$$

Therefore,

$$x_1 = \tan\theta = +16.45194856, +0.93714255; -1.059159498$$

$$\theta = 86.52166587^o, 43.14148763^o; 133.3543492^o$$

$$2\theta = 173.0433317^o; 86.28297527^o; 266.7086985^o$$

$$\tan(2\theta) = -0.122016948; 15.39278891; 17.38909148$$

$$x_2 = -\tan(2\theta) = 0.122016948; -15.39278891; -17.38909148$$

Check,

$$2\bar{x} = \mathrm{x}_1 - \mathrm{x}_2$$

$$= (16.45194856, +0.93714255; -1.059159498) - (0.122016948; -15.39278891; -17.38909148)$$

$$= 16.32993162$$

$$\bar{x} = 8.164965809$$

$$\bar{x} = x_1 - \mathrm{x_M}$$ [Ref. Figure 13]

$$x_{\mathrm{M}} = x_1 - \bar{x}$$

$$= (16.45194856, +0.93714255; -1.059159498) - 8.164965809$$

$$= 8.286982751; -7.227823259; -9.224125307$$

Verification,

$$\bar{x} = \mathrm{x_M} - \mathrm{x}_2$$ [Ref. Figure 13]

$$x_{\mathrm{M}} = x_2 + \bar{x}$$

$$= (0.122016948; -15.39278891; -17.38909148) + 8.164965809$$

$$= 8.286982751; -7.227823259; -9.224125307$$

Where,

$$b = -2a\mathrm{x_M}$$

$$= -2(3)(8.286982751; -7.227823259; -9.224125307)$$

$$= -49.72189651; +43.36693956; +55.34475184$$

$$c = a(\mathrm{x_M})^2 - 200$$

$$= 3(8.286982751; -7.227823259; -9.224125307)^2 - 200$$

$$= 3(68.67408312; +52.24142906; +85.08448768) - 200$$

$$= (206.0222493; +156.7242872; +255.253463) - 200$$

$$= +6.0222493; -43.2757128; +55.253463$$

As such,

$$3\mathrm{x}^2 - 49.72189651x + 6.0222493 = y$$

$$3\mathrm{x}^2 + 43.36693956x - 43.2757128; = y$$

$$3\mathrm{x}^2 + 55.34475184x + 55.253463 = y$$

Check,

$$\frac{b}{a} = -(x_1 + x_2) \qquad \textit{(Ref. small table Section 14.1.2)}$$

$$\frac{-49.72189651}{3} = -[(16.45194856, +0.93714255; -1.059159498) + (0.122016948; -15.39278891; -17.38909148)]$$

$$-16.5739655 = -16.5739655; +14.45564652; +18.44825061$$

$$\frac{43.36693956}{3} = -[(16.45194856, +0.93714255; -1.059159498) + (0.122016948; -15.39278891; -17.38909148)]$$

$$+14.45564652 = -16.5739655; +14.45564652; +18.44825061$$

$$\frac{55.34475184}{3} = -[(16.45194856, +0.93714255; -1.059159498) + (0.122016948; -15.39278891; -17.38909148)]$$

$$+18.44825061 = -16.5739655; +14.45564652; +18.44825061$$

And,

$$\frac{b}{a} = \frac{\tan\theta + \tan^3\theta}{1 - \tan^2\theta} \qquad \textit{(Ref. small table Section 14.1.2)}$$

$$= \frac{x_1 + x_1^{\,3}}{1 - x_1^{\,2}}$$

$$= \frac{16.45194856 + (16.45194856)^3}{1 - (16.45194856)^2}; \frac{0.93714255 + (0.93714255)^3}{1 - (0.93714255)^2}; \frac{-1.059159498 + (-1.059159498)^3}{1 - (-1.059159498)^2}$$

$$= \frac{4469.445117}{-269.6666114}; \frac{1.760175024}{0.121763841}; \frac{-2.24734458}{-0.121818842}$$

$$= -16.5739655; +14.45564652; +18.44825061$$

Table 40 depicts seven *Parabolic Functions* whose *centerlines (Ref. Figure 73) intersect* the *x-axis* at *respective designated* x_M and y_M distances away from the *given low point* coordinate value of $0;-200$.

Such *intersection points* actually represent *displaced origins* since each, lying *directly above* its *respective low point* along the x-axis, possesses its very own $3x^2 - 200 = y$ function.

Accordingly, *Table 40* signifies a ***map*** where *relative movement* along the x-axis becomes charted by associated changes in *describing parabolic functions.*

Figure 73 gives two close-up views of *root sets* for *Series 1* and *Series 6 Parabolic Functions*, respectively.

Table 41 gives the associated *Figure 73* coordinate values.

Table 40. Displaced Origin Mapping of Describing Parabolic Functions.

CONDITIONS x_1	x_2	DESCRIBING PARABOLIC FUNCTIONS	x_M	y_M
$\tan\theta$	$-1/\tan(2\theta)$	$3x^2 - 146.7855003\,x + 1595.498592 = y_{Series1}$ $3x^2 + 48.80591057\,x - 1.498591087 = y_{Series2}$	$+24.46425005$ -8.134318429	-200
$\tan\theta$	$-1/\tan\theta$	$3x^2 + 48.62098315x - 3 = y_{Series3}$ $3x^2 - 48.62098315x - 3 = y_{Series4}$	$+8.103497191$ -8.103497191	-200
$\tan\theta$	$-\tan(2\theta)$	$3x^2 - 49.72189651x + 6.0222493 = y_{Series5}$ $3x^2 + 43.36693956x - 43.2757128 = y_{Series6}$ $3x^2 + 55.34475184x + 55.253463 = y_{Series7}$	$+8.286982751$ -7.227823259 -9.224125307	-200

Figure 73. Mapping of Seven *Describing Parabolic Functions.*

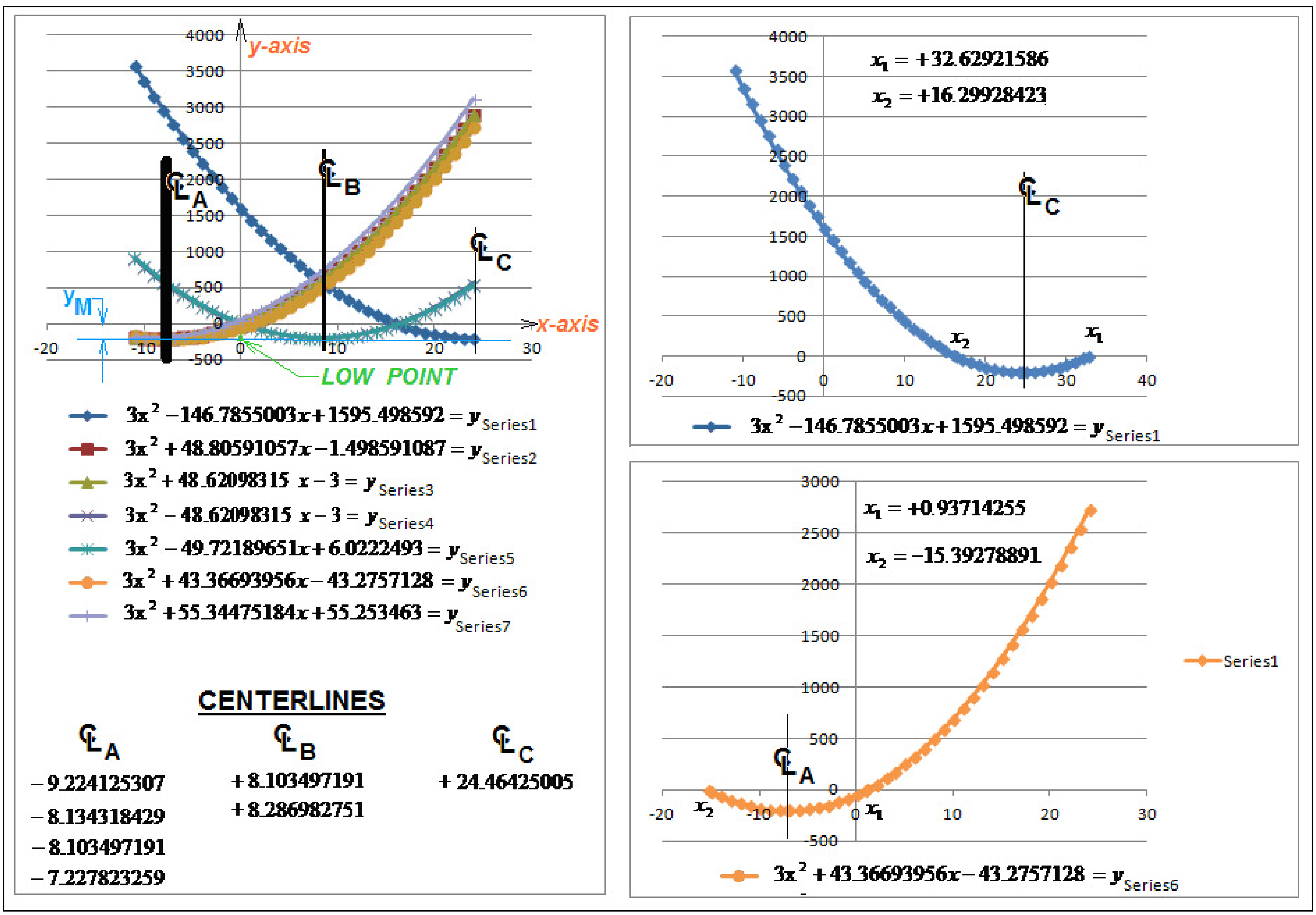


Table 41. *Seven Describing Parabolic Function* Coordinate Values.

x Values	$y_{Series\ 1}$	$y_{Series\ 2}$	$y_{Series\ 3}$	$y_{Series\ 4}$	$y_{Series\ 5}$	$y_{Series\ 6}$	$y_{Series\ 7}$
24	-199.3534152	2897.843263	2891.903596	558.0964044	540.6967331	2725.530837	3111.527507
23	-193.5679149	2708.037352	2702.282612	465.7173876	449.4186296	2541.163897	2915.182755
22	-181.7824146	2524.231441	2518.661629	379.3383707	364.1405261	2362.796958	2724.838003
21	-163.9969143	2346.425531	2341.040646	298.9593539	284.8624226	2190.430018	2540.493252
20	-140.211414	2174.61962	2169.419663	224.580337	211.5843191	2024.063078	2362.1485
19	-110.4259137	2008.81371	2003.79868	156.2013202	144.3062156	1863.696139	2189.803748
18	-74.6404134	1849.007799	1844.177697	93.8223033	83.02811212	1709.329199	2023.458996
17	-32.8549131	1695.201889	1690.556714	37.44328645	27.75000863	1560.96226	1863.114244
16	14.9305872	1547.395978	1542.93573	-12.9357304	-21.52809486	1418.59532	1708.769492
15	68.7160875	1405.590067	1401.314747	-57.31474725	-64.80619835	1282.228381	1560.424741
14	128.5015878	1269.784157	1265.693764	-95.6937641	-102.0843018	1151.861441	1418.079989
13	194.2870881	1139.978246	1136.072781	-128.072781	-133.3624053	1027.494501	1281.735237
12	266.0725884	1016.172336	1012.451798	-154.4517978	-158.6405088	909.1275619	1151.390485
11	343.8580887	898.3664252	894.8308147	-174.8308147	-177.9186123	796.7606224	1027.045733
10	427.643589	786.5605146	783.2098315	-189.2098315	-191.1967158	690.3936828	908.7009814
9	517.4290893	680.754604	677.5888484	-197.5888484	-198.4748193	590.0267432	796.3562296
8	613.2145896	580.9486935	577.9678652	-199.9678652	-199.7529228	495.6598037	690.0114777
7	715.0000899	487.1427829	484.3468821	-196.3468821	-195.0310263	407.2928641	589.6667259
6	822.7855902	399.3368723	396.7258989	-186.7258989	-184.3091298	324.9259246	495.321974
5	936.5710905	317.5309618	315.1049158	-171.1049158	-167.5872333	248.558985	406.9772222
4	1056.356591	241.7250512	239.4839326	-149.4839326	-144.8653367	178.1920454	324.6324704
3	1182.142091	171.9191406	169.8629495	-121.8629495	-116.1434402	113.8251059	248.2877185
2	1313.927591	108.1132301	106.2419663	-88.2419663	-81.42154372	55.45816632	177.9429667
1	1451.713092	50.30731948	48.62098315	-48.62098315	-40.69964721	3.09122676	113.5982148
0	1595.498592	-1.498591087	-3	-3	6.0222493	-43.2757128	55.253463
-1	1745.284092	-47.30450166	-48.62098315	48.62098315	58.74414581	-83.64265236	2.90871116
-2	1901.069593	-87.11041223	-88.2419663	106.2419663	117.4660423	-118.0095919	-43.43604068
-3	2062.855093	-120.9163228	-121.8629495	169.8629495	182.1879388	-146.3765315	-83.78079252
-4	2230.640593	-148.7222334	-149.4839326	239.4839326	252.9098353	-168.743471	-118.1255444
-5	2404.426094	-170.5281439	-171.1049158	315.1049158	329.6317319	-185.1104106	-146.4702962
-6	2584.211594	-186.3340545	-186.7258989	396.7258989	412.3536284	-195.4773502	-168.815048
-7	2769.997094	-196.1399651	-196.3468821	484.3468821	501.0755249	-199.8442897	-185.1597999
-8	2961.782594	-199.9458756	-199.9678652	577.9678652	595.7974214	-198.2112293	-195.5045517
-9	3159.568095	-197.7517862	-197.5888484	677.5888484	696.5193179	-190.5781688	-199.8493036
-10	3363.353595	-189.5576968	-189.2098315	783.2098315	803.2412144	-176.9451084	-198.1940554
-11	3573.139095	-175.3636074	-174.8308147	894.8308147	915.9631109	-157.312048	-190.5388072

The *respective low point* for the *given function* is verified to be:

$$3x_M{}^2 - 200 = y_M$$
$$3(0)^2 - 200 = y_M$$
$$-200 = y_M$$

Note that no y values designated in *Table 38* drop below this value. This is because all of the *describing parabolic functions* appearing in *Figure 73* possess the *same exact low point value*. This is easily verified by substituting designated x_M values into respective *describing parabolic functions*, and thereby noting that for each one:

$$y_M = -200 \ .$$

Because of this affinity, all *describing parabolic functions* are easily verified to bear the same exact shape as the *given function* simply by *sliding* them *mathematically*, in order to allow each to *superimpose* upon the *given function*. This is achieved by means of applying respective *transforms* as follows:

Where,

$$x' = x + x_M$$

$$3x'^2 - 146.7855003x' + 1595.498592 = y'_{Series1}$$
$$3(x + x_M)^2 - 146.7855003(x + x_M) + 1595.498592 = y'_{Series1}$$
$$3x'^2 + 48.80591057x' - 1.498591087 = y'_{Series2}$$
$$3(x + x_M)^2 + 48.80591057(x + x_M) - 1.498591087 = y'_{Series2}$$
$$3x'^2 + 48.62098315x' - 3 = y'_{Series3}$$
$$3(x + x_M)^2 + 48.62098315(x + x_M) - 3 = y'_{Series3}$$
$$3x'^2 - 48.62098315x' - 3 = y'_{Series4}$$
$$3(x + x_M)^2 - 48.62098315(x + x_M) - 3 = y'_{Series4}$$
$$3x'^2 - 49.72189651x' + 6.0222493 = y'_{Series5}$$
$$3(x + x_M)^2 - 49.72189651(x + x_M) + 6.0222493 = y'_{Series5}$$
$$3x'^2 + 43.36693956x' - 43.2757128 = y'_{Series6}$$
$$3(x + x_M)^2 + 43.36693956(x + x_M) - 43.2757128 = y'_{Series6}$$
$$3x'^2 + 55.34475184x' + 55.253463 = y'_{Series7}$$
$$3(x + x_M)^2 + 55.34475184(x + x_M) + 55.253463 = y'_{Series7}$$

Such calculations are afforded in *Table 42* for respective *transformed describing functions*. Notice that values are equal for *all line items*, thereby demonstrating that the curves are of identical shape.

Table 42. Same Shape Verification Chart for *Seven Transformed Functions.*

x Values	$y'_{Series\ 1}$	$y'_{Series\ 2}$	$y'_{Series\ 3}$	$y'_{Series\ 4}$	$y'_{Series\ 5}$	$y'_{Series\ 6}$	$y'_{Series\ 7}$
24	1528	1528	1528	1528	1528	1528	1528
23	1387	1387	1387	1387	1387	1387	1387
22	1252	1252	1252	1252	1252	1252	1252
21	1123	1123	1123	1123	1123	1123	1123
20	1000	1000	1000	1000	1000	1000	1000
19	883	883	883	883	883	883	883
18	772	772	772	772	772	772	772
17	667	667	667	667	667	667	667
16	568	568	568	568	568	568	568
15	475	475	475	475	475	475	475
14	388	388	388	388	388	388	388
13	307	307	307	307	307	307	307
12	232	232	232	232	232	232	232
11	163	163	163	163	163	163	163
10	100	100	100	100	100	100	100
9	43	43	43	43	43	43	43
8	-8	-8	-8	-8	-8	-8	-8
7	-53	-53	-53	-53	-53	-53	-53
6	-92	-92	-92	-92	-92	-92	-92
5	-125	-125	-125	-125	-125	-125	-125
4	-152	-152	-152	-152	-152	-152	-152
3	-173	-173	-173	-173	-173	-173	-173
2	-188	-188	-188	-188	-188	-188	-188
1	-197	-197	-197	-197	-197	-197	-197
0	-200	-200	-200	-200	-200	-200	-200
-1	-197	-197	-197	-197	-197	-197	-197
-2	-188	-188	-188	-188	-188	-188	-188
-3	-173	-173	-173	-173	-173	-173	-173
-4	-152	-152	-152	-152	-152	-152	-152
-5	-5	-5	-5	-5	-5	-5	-5
-6	-2	-2	-2	-2	-2	-2	-2
-7	-3	-3	-3	-3	-3	-3	-3
-8	-8	-8	-8	-8	-8	-8	-8
-9	43	43	43	43	43	43	43
-10	100	100	100	100	100	100	100
-11	163	163	163	163	163	163	163

PROBLEM NUMBER 40 *(Ref. Section 14.1.2)*

GIVEN:

The following *Parabolic Function*:

$$3x^2 - 200 = y$$

LOCATE:

A *series* of *Parabolic Curves* each of which bear the *exact curve shape* and *same low point value* as the *given curve*, but whose *displaced origins* reside upon the circle $x^2 + y^2 = (\bar{x})^2$ at 0°, 45°, 90°, 135°, and 180° locations

SOLUTION:

The *unknown curves* are to be represented by the following *Parabolic* form:

$$3x^2 + bx + c = y$$

Where,

$$\bar{x} = \pm\sqrt{\frac{200}{3}}$$ *[Ref. Problem Number 39]*

$$x_{M1} = \bar{x} = +\sqrt{\frac{200}{3}} \qquad y_{M1} = -200$$

$$x_{M2} = \frac{\sqrt{2}}{2}\bar{x} = +\sqrt{\frac{100}{3}} \qquad y_{M2} = -\frac{\sqrt{2}}{2}\bar{x} - 200 = -\sqrt{\frac{100}{3}} - 200$$

$$x_{M3} = 0 \qquad y_{M3} = -\bar{x} - 200 = -\sqrt{\frac{200}{3}} - 200$$

$$x_{M4} = -\frac{\sqrt{2}}{2}\bar{x} = -\sqrt{\frac{100}{3}} \qquad y_{M2} = -\frac{\sqrt{2}}{2}\bar{x} - 200 = -\sqrt{\frac{100}{3}} - 200$$

$$x_{M5} = -\bar{x} = -\sqrt{\frac{200}{3}} \qquad y_{M5} = -200$$

Such that

$$b = -2a\mathrm{x}_{\mathrm{M}}$$

$$c = a(\mathrm{x}_{\mathrm{M}})^2 - 200$$

$$b_1 = -2a\mathrm{x}_{\mathrm{M}1} = -2(3)\sqrt{\frac{200}{3}} = -\sqrt{2400} \qquad c_1 = a(\mathrm{x}_{\mathrm{M}1})^2 - 200 = 3(\frac{200}{3}) - 200 = 0$$

$$b_2 = -2ax_{M2} = -6\frac{\sqrt{2}}{2}\bar{x} = -\sqrt{1200} \qquad c_2 = a(\mathrm{x}_{\mathrm{M}2})^2 - 200 = 3(\frac{100}{3}) - 200 = -100$$

$$b_3 = -2ax_{M3} = 0 \qquad c_3 = a(\mathrm{x}_{\mathrm{M}3})^2 - 200 = 3(0) - 200 = -200$$

$$b_4 = -2ax_{M4} = +6\frac{\sqrt{2}}{2}\bar{x} = +\sqrt{1200} \qquad c_4 = a(\mathrm{x}_{\mathrm{M}4})^2 - 200 = 3(\frac{100}{3}) - 200 = -100$$

$$b_5 = -2ax_{M5} = +6\bar{x} = +\sqrt{2400} \qquad c_5 = a(\mathrm{x}_{\mathrm{M}5})^2 - 200 = 3(\frac{200}{3}) - 200 = 0$$

Therefore, the five *describing parabolic functions* are as follows:

$$3\mathrm{x}^2 - \sqrt{2400}x = y_{Series1}$$

$$3\mathrm{x}^2 - \sqrt{1200}x - 100 = y_{Series2}$$

$$3\mathrm{x}^2 - 200 = y_{Series3}$$

$$3\mathrm{x}^2 + \sqrt{1200}x - 100 = y_{Series4}$$

$$3\mathrm{x}^2 + \sqrt{2400}x = y_{Series5}$$

Centerlines for these *five parabolic functions are depicted in Figure 74. Each intersects* the *circle (Ref. upper illustration)* at *respective designated* x_M and y_M distances away from the *given low point* coordinate value of $0;-200$.

In the *upper illustration* the *circle* shows clearly because portrayed horizontal distances are equivalent to the same measured vertical distances.

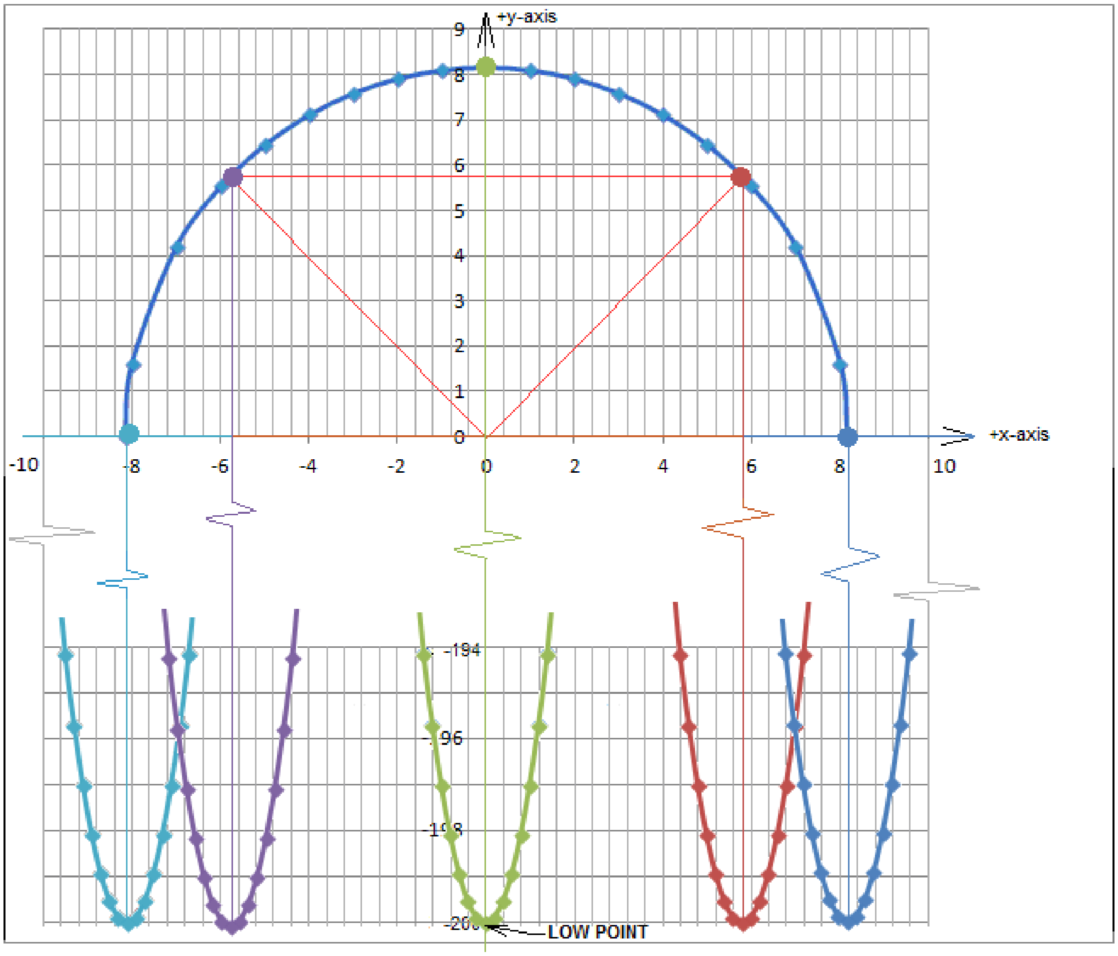
This is not so for the *lower illustration* because the vertical distances displayed are much greater than the horizontal ones. Hence, the *circle* would not show up as such in the lower illustration.

Intersection *points* along the circle *(Ref. upper illustration)* actually represent *displaced origins* since each, lying *directly above* its *respective low point* along the x-axis, possesses its very own $3x^2 - 200 = y$ function.

Accordingly, these *five describing parabolic functions* designate a ***map*** where *relative movement* along the x-axis becomes charted by their associated changes.

Table 43 gives the associated *Figure 74* coordinate values.

Figure 74. Mapping of Five *Describing Parabolic Functions.*



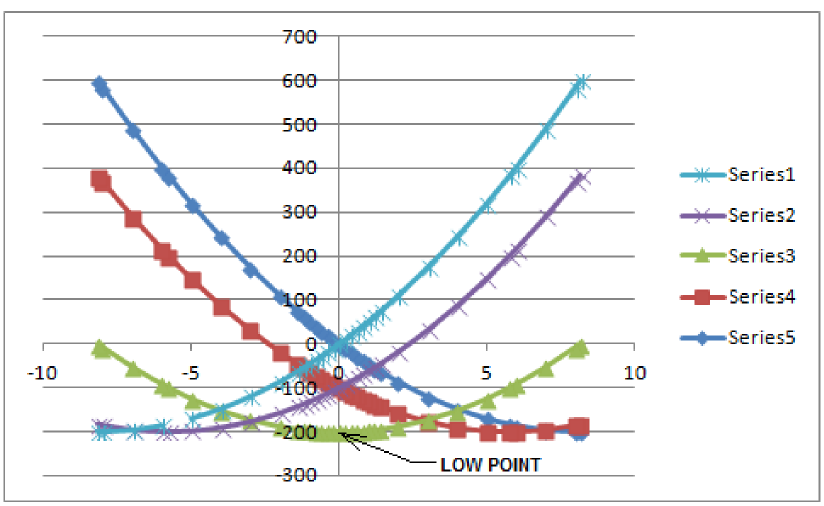


Table 43. ***Five Describing Parabolic Function*** **Coordinate Values.**

x Values	$y_{Series\ 1}$	$y_{Series\ 2}$	$y_{Series\ 3}$	$y_{Series\ 4}$	$y_{Series\ 5}$
24	552.2449234	796.6156124	1528	2459.384388	2903.755077
23	460.2347182	690.2566286	1387	2283.743371	2713.765282
22	374.2245131	589.8976447	1252	2114.102355	2529.775487
21	294.2143079	495.5386609	1123	1950.461339	2351.785692
20	220.2041028	407.179677	1000	1792.820323	2179.795897
19	152.1938977	324.8206932	883	1641.179307	2013.806102
18	90.18369252	248.4617093	772	1495.538291	1853.816307
17	34.17348738	178.1027255	667	1355.897275	1699.826513
16	-15.83671776	113.7437416	568	1222.256258	1551.836718
15	-59.8469229	55.38475775	475	1094.615242	1409.846923
14	-97.85712804	3.0257739	388	972.9742261	1273.857128
13	-129.8673332	-43.33320995	307	857.33321	1143.867333
12	-155.8775383	-83.6921938	232	747.6921938	1019.877538
11	-175.8877435	-118.0511777	163	644.0511777	901.8877435
10	-189.8979486	-146.4101615	100	546.4101615	789.8979486
9	-197.9081537	-168.7691454	43	454.7691454	683.9081537
8	-199.9183589	-185.1281292	-8	369.1281292	583.9183589
7	-195.928564	-195.4871131	-53	289.4871131	489.928564
6	-185.9387692	-199.8460969	-92	215.8460969	401.9387692
5	-169.9489743	-198.2050808	-125	148.2050808	319.9489743
4	-147.9591794	-190.5640646	-152	86.5640646	243.9591794
3	-119.9693846	-176.9230485	-173	30.92304845	173.9693846
2	-85.97958972	-157.2820323	-188	-18.7179677	109.9795897
1	-45.98979486	-131.6410162	-197	-62.35898385	51.98979486
0	0	-100	-200	-100	0
-1	51.98979486	-62.35898385	-197	-131.6410162	-45.98979486
-2	109.9795897	-18.7179677	-188	-157.2820323	-85.97958972
-3	173.9693846	30.92304845	-173	-176.9230485	-119.9693846
-4	243.9591794	86.5640646	-152	-190.5640646	-147.9591794
-5	319.9489743	148.2050808	-125	-198.2050808	-169.9489743
-6	401.9387692	215.8460969	-92	-199.8460969	-185.9387692
-7	489.928564	289.4871131	-53	-195.4871131	-195.928564
-8	583.9183589	369.1281292	-8	-185.1281292	-199.9183589
-9	683.9081537	454.7691454	43	-168.7691454	-197.9081537
-10	789.8979486	546.4101615	100	-146.4101615	-189.8979486
-11	901.8877435	644.0511777	163	-118.0511777	-175.8877435

The *respective low point* for the *given function* is verified to be:

$$3{x_M}^2 - 200 = y_M$$
$$3(0)^2 - 200 = y_M$$
$$-200 = y_M$$

Note that no y values designated in *Table 43* drop below this value. This is because all of the *describing parabolic functions* appearing in *Figure 74* possess the *same exact low point value*. This is easily verified by substituting designated x_M values into respective *describing parabolic functions*, and thereby noting that for each one:

$$y_M = -200\ .$$

Because of this affinity, all *describing parabolic functions* are easily verified to bear the same exact shape as the *given function* simply by *sliding* them *mathematically*, in order to allow each to *superimpose* upon the *given function*. This is achieved by means of applying respective *transforms* as follows:

Where,

$$x' = x + x_M$$

$$3x'^2 - \sqrt{2400}x' = y'_{Series1}$$
$$3(x + x_M)^2 - \sqrt{2400}(x + x_M) = y'_{Series1}$$
$$3x'^2 - \sqrt{1200}x' - 100 = y'_{Series2}$$
$$3(x + x_M)^2 - \sqrt{1200}(x + x_M) - 100 = y'_{Series2}$$
$$3x'^2 - 200 = y'_{Series3}$$
$$3(x + x_M)^2 - 200 = y'_{Series3}$$
$$3x'^2 + \sqrt{1200}x' - 100 = y'_{Series4}$$
$$3(x + x_M)^2 + \sqrt{1200}(x + x_M) - 100 = y'_{Series4}$$
$$3x'^2 + \sqrt{2400}x' = y'_{Series5}$$
$$3(x + x_M)^2 + \sqrt{2400}(x + x_M) = y'_{Series5}$$

Such calculations are afforded in *Table 44* for respective *transformed describing functions*. Notice that values are equal for *all line items*, thereby demonstrating that the curves are of identical shape.

Table 44. Same Shape Verification Chart for *Five Transformed Functions.*

x Values	$y'_{Series\ 1}$	$y'_{Series\ 2}$	$y'_{Series\ 3}$	$y'_{Series\ 4}$	$y'_{Series\ 5}$
24	1528	1528	1528	1528	1528
23	1387	1387	1387	1387	1387
22	1252	1252	1252	1252	1252
21	1123	1123	1123	1123	1123
20	1000	1000	1000	1000	1000
19	883	883	883	883	883
18	772	772	772	772	772
17	667	667	667	667	667
16	568	568	568	568	568
15	475	475	475	475	475
14	388	388	388	388	388
13	307	307	307	307	307
12	232	232	232	232	232
11	163	163	163	163	163
10	100	100	100	100	100
9	43	43	43	43	43
8	-8	-8	-8	-8	-8
7	-53	-53	-53	-53	-53
6	-92	-92	-92	-92	-92
5	-125	-125	-125	-125	-125
4	-152	-152	-152	-152	-152
3	-173	-173	-173	-173	-173
2	-188	-188	-188	-188	-188
1	-197	-197	-197	-197	-197
0	-200	-200	-200	-200	-200
-1	-197	-197	-197	-197	-197
-2	-188	-188	-188	-188	-188
-3	-173	-173	-173	-173	-173
-4	-152	-152	-152	-152	-152
-5	-5	-5	-5	-5	-5
-6	-2	-2	-2	-2	-2
-7	-3	-3	-3	-3	-3
-8	-8	-8	-8	-8	-8
-9	43	43	43	43	43
-10	100	100	100	100	100
-11	163	163	163	163	163